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SOME INTERESTING PROCESSES ARISING AS HEAVY TRAFFIC LIMITS IN AN M/M/∞ STORAGE PROCESS

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Recent works by Newell [13] and Coffman et al. [2] have studied a queuing or storage model which is most easily visualized as the process of parking cars in a parking lot where customers park as close as they can to some fixed point. This paper describes several space-time processes which arise as heavy-traffic limits in this model, and which seem interesting in their own right.

M/M/∞ queue * dynamic storage allocation * heavy-traffic * infinite-dimensional diffusion * extremal properties of stationary processes

1. Introduction

Imagine a supermarket with a parking lot with parking spaces labeled 1, 2, 3, ... A simple and natural model for the arrival, parking and eventual departure of cars is the following.

(a) Cars arrive as a Poisson process of rate λ.

(b) Each arriving car parks in the lowest-numbered available space.

(c) Each car remains for a random time with exponential (1) distribution, then departs.

This model arises in several other contexts. It can be interpreted as an M/M/∞ queue where the servers are ranked, and each arriving customer (car) is served by the lowest-ranked free server (parking space). From that viewpoint it is studied in a recent monograph of Newell [13], which is a masterpiece of classical-style applied probability. Coffman et al. [2] study it as a model of storage allocation in computer memory.

The total number of parked cars evolves precisely as the M/M/∞ queue, which is readily analyzed. The questions of interest for this model concern the spatial distribution of parked cars. The model describes a space-time process, Markov in the time variable, whose (time-) stationary distribution is a complicated spatial process. In the heavy-traffic ($\lambda \to \infty$) limit one would expect to find some limiting space-time process, whose stationary distribution would be a spatial process which could be used to approximate the distribution of where cars are parked. It turns

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out that there are four different limiting processes, corresponding to different regions of the parking lot:

(a) Spaces 1 to $(1-\varepsilon)\lambda$: the "geometric process" (Section 2).

(b) Spaces around $\lambda - 0(\lambda^b)$, $\frac{1}{2} < b < 1$: the "exponential process" (Section 3).

(c) Spaces $\lambda \pm 0(\lambda^{1/2})$: the "truncated Normal process" (Section 4).

(d) Spaces around $\lambda + \sqrt{2\lambda \log\log \lambda}^{1/2}$: an 'extremal process" (Section 5).

The purpose of this paper is to describe these limit processes.

Heavy traffic limit theory for queues is a well-developed subject: see Whitt [16] and Iglehart [8] for surveys, and Borovkov [1] for a detailed treatment. We will sketch proofs that our processes are indeed the heavy-traffic limits of the parking model, but these weak convergence arguments (some of which are implicit in [13]) are *not* the main point of the paper. Rather, we wish to present these processes as interesting *examples* for stochastic process theory. Interesting because they arise naturally, they are sufficiently structured to be partly tractable yet sufficiently complicated to make explicit calculations hard, and because their analysis illustrates known general techniques and suggests new general problems. Relevant techniques include coupling, excursion theory, the theory of semi-local maxima of stationary processes and the theory of priority queues. New general problems (Section 6) are the approximate independence of subprocesses which run on incompatible time-scales; structure theory for function-valued diffusions; and self-similarity and convergence for weak limits of parametric families of processes.

We first record some notation and elementary properties of the basic parking lot process, which are contained in [2, 13]. The state of the process at time t can be represented as a random vector $(S(1,t), S(2,t), S(3,t), \ldots; S(\infty,t))$ where $S(m,t)$ is the number of occupied spaces amongst spaces 1 through m; and $S(\infty,t)$ is the total number of occupied spaces. The process $S(\infty,t)$, which records the total number of parked cars without regard for their position, is just the familiar $M/M/\infty$ queue. The stationary distribution $S(\infty)$ is the Poisson (λ) distribution, and starting from any initial value $S(\infty,0)$ the distributions $S(\infty,t)$ converge as $t\to\infty$ to $S(\infty)$. It is easy to deduce (e.g. by the coupling arguments of Section 2) that the whole process $((S(1,t), S(2,t), \ldots; S(\infty,t))$ has a stationary distribution $(S(1), S(2), \ldots; S(\infty))$, to which it converges in distribution as $t\to\infty$ from any initial configuration.

For fixed m, the 1-dimensional process $S(m,t)$ evolves as the Markov chain on $\{0,1,\ldots,m\}$ with transition rates

$$i \to i+1: \quad \text{rate } \lambda,$$

$$i \to i-1: \quad \text{rate } i.$$

This is the $M/M/m$ queue where arrivals are lost if all servers are busy. The stationary distribution is

$$S(m) \text{ has Poisson}(\lambda) \text{ distribution truncated to } \{0,1,\ldots,m\}. \tag{1.1}$$

For fixed $k<m$, the bivariate process $(S(k,t), S(m,t))$ is again a Markov chain whose transition rates are easily written down. Generating function techniques yield

an expression [13, p. 4] for the stationary distribution $(S(k), S(\infty)-S(k))$, but this expression is complicated.

Some information can be derived from (1.1). Let L be the stationary distribution of the lowest-numbered unoccupied space. Then $L=m$ iff $S(m-1)=m-1$ and $S(m)\neq m$. So

$$P(L=m)=P(S(m-1)=m-1)-P(S(m)=m)$$
$$=\frac{P(N(\lambda)=m-1)}{P(N(\lambda)\leq m-1)}-\frac{P(N(\lambda)=m)}{P(N(\lambda)\leq m)} \tag{1.2}$$

where $N(\lambda)$ indicates the Poisson(λ) distribution. Next, let $\pi(m)$ be the stationary probability that space m is occupied. If we watch space m then we see transitions

occupied $\to$ unocuppied: rate 1,

unoccupied $\to$ occupied: rate λ if $L=m$, rate 0 otherwise.

The ergodic argument implies $1\cdot\pi(m)=\lambda\cdot P(L=m)$, so

$$\pi(m)=\lambda\left\{\frac{P(N(\lambda)=m-1)}{P(N(\lambda)\leq m-1)}-\frac{P(N(\lambda)=m)}{P(N(\lambda)\leq m)}\right\}. \tag{1.3}$$

One can study the heavy-traffic behavior by taking limits in these exact expressions; such analytic arguments are the subject of [13].

We end this introduction with an intuitive description of how the heavy-traffic process varies over the different regions of the parking lot. Write $\bar{S}(m,t)=m-S(m,t)$ for the number of empty spaces amongst spaces 1 through m.

Case (a). Let $m=x\lambda$, for fixed $x<1$. Empty spaces are removed by arriving cars (rate λ) and created by departing cars (rate about $x\lambda$, since for this purpose there are a negligible number of empty spaces). Thus $\bar{S}(m,t)$ evolves like an M/M/1 queue with arrival rate $x\lambda$ and service rate λ. So the mean number of empty spaces $E\bar{S}(m)\approx 1/(1-x)$, which stays finite as $\lambda\to\infty$. Varying x gives a space-time process of empty parking spaces, which in the heavy-traffic limit becomes a coupled family of M/M/1 queues with varying arrival rates, the *geometric process* (Section 2).

Case (b). Recall that the heavy-traffic limit of M/M/1 queues is Brownian motion with drift reflected at 0. From (a), we guess that if $m=m(\lambda)$ satisfies $m/\lambda\uparrow 1$ slowly, then $E\bar{S}(m)\to\infty$ and the normalized process $\bar{S}(m,t)$ approximates this Brownian motion with drift. It turns out this approximation works for $m(\lambda)\sim\lambda-a\lambda^b$ $(\frac{1}{2}<b<1)$, and that by varying a we get a coupled family of Brownian motions with varying drifts as the heavy-traffic limit, the *exponential process* (Section 3).

Case (c). For $m(\lambda)=\lambda\pm 0(\lambda^{1/2})$ the behavior changes, because the total number $S(\infty,t)$ of parked cars fluctuates over this region. There are typically $O(\lambda^{1/2})$ empty spaces to the left of m, and in discussing the departure rate of cars from this region we can no longer neglect the empty spaces. As the number $\bar{S}(m,t)$ of empty spaces increases, the drift rate of $\bar{S}(m)$ (= departure rate of cars − arrival rate of cars) becomes more negative. In the heavy-traffic limit the normalized process $\bar{S}(m,t)$ approaches an Ornstein–Uhlenbeck process (Section 4).

Case (d). For $m > \lambda + 0(\lambda^{1/2})$ let $\hat{S}(m, t) = S(\infty, t) - S(m, t)$ be the number of cars parked in the region to the right of m. Cars park in this region only on the rare occasions when $S(\infty, t) \geq m$. When this happens, a bunch of $O(\lambda^{1/2})$ cars park in that region, and the last car in this bunch to depart stays about time $\log \lambda^{1/2}$. Thus there is a critical value of m for which the rate of distinct upcrossings of $S(\infty, t)$ over m is $1/\log \lambda^{1/2}$. For m around this critical value, there is an extremal process which describes when $S(\infty, t)$ goes above m and how many cars park on such occasions, and this extremal process controls $\hat{S}(m, t)$ (Section 5).

2. The geometric process

Consider the set S of sequences $0 \leq x_1 < x_2 < x_3 < \cdots < 1$ such that $\#\{n: x_n \leq y\} < \infty$ for $y < 1$. Such a sequence can be described via the counting function $(n(x); 0 \leq x < 1)$:

$$n(x) = \#\{k: x_k \leq x\}.$$

A random element of S can be written as a sequence (X_k) or as a counting process $(N(x), 0 \leq x < 1)$.

Consider the following S-valued process. Picture the x's as randomly-positioned points on $[0, 1)$. At the times of a Poisson process (rate 1), a new point is created and placed at a uniform random position in $[0, 1)$. At the times of an independent Poisson process (rate 1), the leftmost point present is destroyed. Write $(N(x, t); 0 \leq x < 1)$ for the configuration at time t. Call this the *geometric process.*

For fixed x, the process $N(x, t)$, $t \geq 0$, evolves as the M/M/1 queue with arrival rate x and service rate 1. This has stationary distribution $N(x)$ with the geometric distribution

$$P(N(x) = i) = (1 - x)x^i, \quad i = 0, 1, 2, \ldots, \tag{2.1}$$

and $N(x, t)$ converges in distribution as $t \to \infty$ to $N(x)$.

Proposition 2.2. *The geometric process has a stationary distribution* $(N(x); 0 \leq x < 1)$ *whose marginals satisfy* (2.1). *From any initial distribution, the process* $N(\cdot, t)$ *converges in distribution to* $N(\cdot)$ *as* $t \to \infty$.

Technical note. S has a natural topology: start with the usual Skorohod topology on $D[0, \infty)$ (see [14]), map $[0, \infty)$ to $[0, 1)$ to get the Skorohod topology on $D[0, 1)$, and regard S as a subset of $D[0, 1)$. For each t, $N(\cdot, t)$ is a random element of S, and Proposition 2.2 asserts weak convergence of their distributions. The result is rather obvious, but we write out the details as a simple illustration of coupling arguments.

Proof. Fix $x_0 < 1$ and consider the process restricted to $[0, x_0]$. This process regenerates every time $N(x_0, t)$ hits 0, and from the M/M/1 nature of $N(x_0, t)$ the mean time between regenerations is finite; this implies the existence of a stationary distribution. Now consider two initial configurations $n(x)$, $\hat{n}(x)$. There is a natural construction of two processes $N(x, t)$, $\hat{N}(x, t)$ starting with the specified initial configurations: use the same time-space Poisson process of creation of points for both processes, and the same Poisson process of times of destroying points in both processes. This construction has the property:

$$\text{if} \quad N(x, t_0) \geq [=] \hat{N}(x, t_0) \quad \text{then} \quad N(x, t) \geq [=] \hat{N}(x, t) \text{ for } t \geq t_0.$$

Supposing $n(x_0) \geq \hat{n}(x_0)$, say, then by considering the first time that $N(x_0, t) = 0$ we see that

$$N(\cdot, t) = \hat{N}(\cdot, t) \quad \text{on } [0, x_0] \text{ for sufficiently large } t. \tag{2.3}$$

Similarly, for any initial distributions $N(x, 0)$, $\hat{N}(x, 0)$ we can construct a coupling satisfying (2.3). By taking $\hat{N}(x, 0)$ to be the the stationary distribution $N(x)$, (2.3) implies $N(\cdot, t)$ converges in distribution to $N(\cdot)$ when restricted to $[0, x_0]$. Since x_0 is arbitrary, this establishes the Proposition.

Now consider the parking lot process $S(m, t)$ with arrival rate λ. Define

$$M_\lambda(x, t) = [x\lambda] - S([x\lambda], t/\lambda), \quad 0 \leq x < 1, t \geq 0. \tag{2.4}$$

This means we are first slowing down time by a factor λ (so arrival rate becomes 1 and mean sojourn time becomes λ), and then $M(x, t)$ counts the number of *empty* parking spaces at time t amongst spaces 1 through $x\lambda$.

Proposition 2.5. *As $\lambda \to \infty$ the space-time process $(M_\lambda(x, t); 0 \leq x < 1, t \geq 0)$ converges in distribution to the geometric process $(N(x, t);\ 0 \leq x < 1,\ t \geq 0)$, provided that $M_\lambda(x, 0)$ converges to $N(x, 0)$. In particular, the stationary distribution $(M_\lambda(x);$ $0 \leq x < 1)$ converges in distribution to the stationary distribution $(N(x);\ 0 \leq x < 1)$.*

Technical note. For fixed t a path $N(\cdot, t)$ is an element of S, and so the map $t \to N(\cdot, t)$ can be considered an element of $D([0, \infty), S) = \hat{D}$. The space-time processes are random elements of $\hat{D}$.

Remark. For the stationary distributions, straightforward calculus shows that (1.1) implies

$$M_\lambda(x) \xrightarrow{\mathcal{D}} N(x) \text{ as } \lambda \to \infty, \quad x \text{ fixed}. \tag{2.6}$$

The point of Proposition 2.5 is to "explain" and amplify this result: for large λ, $M_\lambda(x)$ has approximately geometric marginals because the process $M_\lambda(x, t)$ evolves approximately like the geometric process which has exactly geometric stationary marginals.

Sketch of proof. Fix λ. Given the geometric process $N(x, t)$ and an initial configuration $S(m, 0)$, $1 \leq m \leq [\lambda]$ for the parking lot process restricted to spaces 1 through $[\lambda]$, we can construct a version of the restricted parking lot process $S(m, t)$ (with time slowed by a factor λ) via the following rules.

(a) Whenever the geometric process creates a point, (x, t) say, we make the car (if any) at space $[x\lambda]$ depart at time t.

(b) Whenever the geometric process removes its leftmost point, at time t say, we make a new car arrive and park in the leftmost available space.

The purpose of this construction is to try to match points of the geometric process with empty spaces in the parking lot process. Write $\bar{S}(m, t) = m - S(m, t)$ for the number of empty spaces in 1 through m. Fix m_0. If

$$\bar{S}(m, t_0) = N(m/\lambda, t_0) \quad \text{for all } m \leq m_0$$

then, for all $t \geq t_0$,

(c) $\bar{S}(m, t) \leq N(m/\lambda)$ for all $m \leq m_0$

and moreover there is equality in (c) up until the first time $T > t_0$ that the geometric process has two point in some interval $(m/\lambda, (m+1)/\lambda)$, $m < m_0$.

Now fix $x_0 < 1$. Start the geometric process with 0 points, and start the parking lot process with spaces 1 through λ full, so (c) holds for all $t \geq 0$ with $m_0 = [\lambda x_0]$. Consider the times t at which $N(x_0, t)$ hits 0; as in the previous proof, these are regeneration times for the geometric process restricted to $[0, x_0]$. Let $q = q(x_0, \lambda)$ be the chance that there is strict inequality in (c) sometime between one regeneration and the next. Then $q \to 0$ as $\lambda \to \infty$ by the remark below (c). Thus as $\lambda \to \infty$ the proportion of time that there is inequality in (c) tends to 0; the rest of the proof is straightforward.

Priority queues. It turns out that the geometric process occurs in a quite different context. Consider an M/M/1 queue with arrival rate 1 and service rate 1. Suppose that each arrival is given a priority number, distributed uniformly on $[0, 1)$, and suppose that the server directs his attention to the customer present who has the lowest priority number (interrupting service to other customers when a customer with lower priority number arrives). Then the process whose value at time t is the set of priority numbers of customers present is just the geometric process.

Classical queing theorists have studied priority queues where there are k priority classes, and proportion p_i of customers are in class i. For such a process, with arrival rate $\lambda < 1$, the stationary distribution of the number of customers present in the various priority classes is just $(N(x_1), N(x_2) - N(x_1), \ldots, N(x_k) - N(x_{k-1}))$, where $x_i - x_{i-1} = \lambda p_i$ and $(N(x);\ 0 \leq x < 1)$ is the stationary distribution of the geometric process. Classical results about such priority queues can be found in Jaiswal [10, IV.7]. The geometric process provides a conceptually elegant way of looking at such priority queues.

In the rest of this section we record some facts about the stationary distribution $(N(x);\ 0 \le x < 1)$. We were motivated by the question: can the extra structure of the geometric process be used to give simpler results about priority queues? The answer seems to be no and yes. A classical result in priority queues (Proposition 2.18) gives the generating function of $(N(x), N(y)-N(x))$, and we cannot improve on its derivation. No useful explicit form of this joint distribution is known. For simpler quantities such as $EN(x)(N(y)-N(x))$ we shall write out more explicit results (2.11–2.17). These are not essentially new results, since they could of course be deduced from the classical generating function result (although some of the results refer to structure of the geometric process not present in the classical priority queue model). Our aim is to give a more direct derivation of these explicit results, relying on Lemma 2.10 below.

Let us start by repeating (2.1):

$$P(N(x)=i)=(1-x)x^i, \quad i \ge 0.$$

Let X_k be the position of the kth point. Then

$$P(X_k \le x) = P(N(x) \ge k) = x^k,$$

so X_k has density

$$f_{X_k}(x) = kx^{k-1}, \quad 0 \le x < 1. \tag{2.7}$$

Note in particular that X_1 is uniform on $[0, 1)$. If L_k is the position of the kth empty space in the storage process,

$$(L_1/\lambda, L_2/\lambda, \ldots) \xrightarrow{\mathcal{D}} (X_1, X_2, \ldots) \quad \text{as } j \to \infty,$$

and so in particular L_1 is approximately uniform on $[0, \lambda]$ for large λ.

Let π be the intensity function for N, defined by $\pi(y)\,\mathrm{d}y = P(\text{some point of } N \text{ in } (y, y+\mathrm{d}y))$. Then

$$\pi(y) = \frac{\mathrm{d}}{\mathrm{d}y}(EN(y)) = (1-y)^{-2}. \tag{2.8}$$

The corresponding heavy-traffic limit result is

$$1-\pi_\lambda(m) \sim \lambda^{-1}(1-y)^{-2} \quad \text{as } \lambda \to \infty,\ m/\lambda \to y < 1 \tag{2.9}$$

where $\pi_\lambda(m)$ is the stationary probability that space m is occupied in the parking lot process (1.3).

The fundamental explicit result is

Lemma 2.10. *Let $0 < x < y < 1$, $j \ge 0$. Then $P(N(x)=j$, some point in $(y, y+\mathrm{d}y))$*

$$= (1-x)x^j\left\{\frac{1}{(1-y)^2} + \frac{j}{1-x} - \frac{x}{(1-x)^2}\right\}\mathrm{d}y.$$

This has a nice probabilistic proof, which we give later. One can read off the following.

Corollary

$$P(N(x)\geq j,\ \textit{some point in } (y,y+\mathrm{d}y))=x^j\left\{\frac{1}{(1-y)^2}+\frac{j}{1-x}\right\}\mathrm{d}y. \tag{2.11}$$

$$P(X_j\in(x,x+\mathrm{d}x),\ \textit{some point in } (y,y+\mathrm{d}y))$$
$$=jx^{j-1}\left\{\frac{1}{(1-y)^2}+\frac{j}{1-x}+\frac{x}{(1-x)^2}\right\}\mathrm{d}x\,\mathrm{d}y. \tag{2.12}$$

$$E(N(x);\ \textit{some point in } (y,y+\mathrm{d}y))=\frac{x}{1-x}\left\{\frac{1}{(1-y)^2}+\frac{1}{(1-x)^2}\right\}\mathrm{d}y. \tag{2.13}$$

The joint intensity function, defined by

$$\pi(x,y)\,\mathrm{d}x\,\mathrm{d}y=P(\text{some point in } (x,x+\mathrm{d}x),\text{ some point in } (y,y+\mathrm{d}y)),$$

is given by

$$\pi(x,y)=\frac{1}{(1-x)^2}\left\{\frac{1}{(1-y)^2}-\frac{2}{1-x}+\frac{3}{(1-x)^2}\right\}, \tag{2.14}$$

$$E(N(y)-N(x)|N(x)=j)=\frac{y-x}{1-x}\left\{\frac{1}{1-y}+j-\frac{x}{1-x}\right\}, \tag{2.15}$$

$$E(N(y)-N(x))N(x)=\frac{x(1-y)}{1-x}\left\{\frac{1}{(1-y)^2}-\frac{1}{(1-x)^2}\right\}, \tag{2.16}$$

$$E(N(y)-N(x)|X_j=x)=\frac{y-x}{1-x}\left\{\frac{1}{1-y}+j+\frac{x}{1-x}\right\}. \tag{2.17}$$

These results give the correlation structure of $(N(x))$, but the joint distributions lie deeper, and to obtain them it seems necessary to resort to tedious generating function arguments. The key idea is that, if we watch the process $N(y,t)-N(x,t)$ only at times t when $N(x,t)=0$ then we see a process on $\{0,1,2,\dots\}$ which is skip-free downwards, and whose stationary distribution can therefore be explicitly found in generating function form.

Proposition 2.18. *Fix $0<x<y<1$. Let $G(\theta,\rho)=E\theta^{N(x)}\rho^{N(y)-N(x)}$. Then*

$$G(\theta,\rho)=(1-y)\,\frac{\rho^{-1}-1+(\theta^{-1}-\rho^{-1})/Q(\rho)}{\theta^{-1}-1-x(1-\theta)-(y-x)(1-\rho)}$$

where $Q(\rho)=1+\frac{1}{2}\rho(1-\rho)^{-1}\{1-y+(y-x)\rho-\sqrt{\{1+y-\rho(y-x)\}^2-4x}\}$.

Proposition 2.19. *(X_1,X_2) has joint density*

$$f(x_1,x_2)=2(1+x_2-2x_1)\{(1+x_2)^2-4x_1\}^{-3/2},\quad 0<x_1<x_2<1.$$

We omit the details. Proposition 2.18 is essentially [10, eq. IV.3.15].

Proof of Lemma 2.10. Fix $x_0 < y$. Consider the stationary geometric process $N(x, t)$ run for $-\infty < t < \infty$. Let $U_1, D_1, U_2, D_2, \ldots$ be the times $t \geq 0$ of jumps of $N(x_0, t)$ from 0 to 1, from 1 to 0, from 0 to 1, and so on. Write $N^*(\cdot)$ for the distribution of $N(\cdot, 0)$ given $N(x_0, 0) = 0$. Then

(i) $N(\cdot, U_i-)$ has distribution N^* ("Poisson arrivals see the stationary distribution");

(ii) the segment $(N(x_0, t);\ U_i \leq t < D_i)$ is independent of $N(\cdot, U_i-)$.

Let $U_{(t)} = \max(U_i\colon U_i \leq t)$ and $U_0 = \max(u < 0\colon N(x_0, u) = 0)$. By ergodicity the distribution $(N(\cdot, U_0-), U_0, N(x_0, 0))$ is the limiting empirical distribution of $(N(\cdot, U_{(t)}-), t - U_{(t)}, N(x_0, t))$ as $t \to \infty$; and from the independence property (ii) we can deduce

(iii) $N(\cdot, U_0-)$ is independent of $(U_0, N(x_0, 0))$.

Now define $\phi(j)$ by

$$\phi(j)\,\mathrm{d}y = P(\text{some point of } N(\cdot, 0) \text{ in } (y, y+\mathrm{d}y) | N(x_0, 0) = j).$$

Then

$$\begin{aligned}\phi(j)\,\mathrm{d}y &= P(\text{some point of } N(\cdot, -U_0) \text{ in } (y, y+\mathrm{d}y) | N(x_0, 0) = j)\\ &\quad + P(\text{some point created in } (y, y+\mathrm{d}y)\\ &\quad \text{during } (-U_0, 0) | N(x_0, 0) = j)\\ &= \phi(0)\,\mathrm{d}y + E(-U_0 | N(x_0, 0) = j)\,\mathrm{d}y,\end{aligned}$$

the first because of (iii) and (i), and the second because the creation of points in $(y, y+\mathrm{d}y)$ is independent of $N(x_0, \cdot)$. Now the process $N(x_0, t)$ is time-reversible, so $E(-U_0 | N(x_0, 0) = j)$ is just the mean first passage time from j to 0, which works out as $j/(1-x)$. Thus

(iv) $\phi(j) = \phi(0) + j/(1-x)$.

But the intensity function $\pi(y)$ satisfies

$$\pi(y) = \sum_{i \geq 0} \phi(i) P(N(x) = i) = \sum_{i \geq 0} \{\phi(0) + i/(1-x)\}(1-x)x^i.$$

Using (2.8) we can solve for $\phi(0)$, and then (iv) gives the result.

3. The exponential process

Fix $a < 0$. Let $Y(a, t)$, $t \geq 0$, be Brownian motion with drift a and variance 2, confined to $[0, \infty)$ by a reflecting boundary at 0. In the language of stochastic differential equations we can write

(i) $\mathrm{d}Y(a, t) = a\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B(t)$ on $\{Y(a, t) > 0\}$, (3.1)

(ii) $Y(a,\cdot)$ is reflecting at 0

where $B(t)$ is standard Brownian motion, $B(0)=0$, and where (ii) can be said symbolically using local time. It is well known that $Y(a,t)$ has a stationary distribution $Y(a)$ such that

$$Y(a) \text{ has exponential } (-a) \text{ distribution.} \tag{3.2}$$

By varying a, we can construct a space-time process $Y(a,t)$ which we call the *exponential process.*

Proposition 3.3. *Let* $(Y(a,0);\ a<0)$ *be positive, continuous, increasing and independent of* $B(t)$. *Then there exists a process* $(Y(a,t);\ a<0,\ t\geq 0)$ *such that*

(i) *for each* a, $(Y(a,t);\ t\geq 0)$ *is a reflecting Brownian motion satisfying* (3.2);

(ii) *the sample paths are jointly continuous in* (a,t), *and increasing in* a;

(iii) *as* $t\to\infty$ *the processes* $(Y(a,t);\ a<0)$ *converge in distribution to a stationary distribution* $(Y(a);\ a<0)$;

(iv) *the stationary distribution* $(Y(a);\ a<0)$ *satisfies*

(a) *the paths* $a\to Y(a)$ *are continuous, increasing*;

(b) $\lim_{a\to-\infty} Y(a)=0$; $\lim_{a\to 0} Y(a)=\infty$;

(c) *for each* a, $Y(a)$ *has exponential* $(-a)$ *distribution.*

Proof. Write $X(a,t)=at+\sqrt{2}B(t)$, so that $X(a,\cdot)$ is unconstrained Brownian motion with drift. Let

$$Y(a,t)=X(a,t)-\min_{0\leq u\leq t} X(a,u). \tag{3.4}$$

Here $Y(a,0)\equiv 0$. It is well known [7] that for fixed a, $Y(a,t)$ is reflecting Brownian motion (3.1). Properties (ii) are easily checked. Now consider $B(t)$, and hence $X(a,t)$, extended to $-\infty<t<\infty$, and consider

$$Y^*(a,t)=X(a,t)-\min_{-\infty<u\leq t} X(a,u). \tag{3.5}$$

Then the process $Y^*(a,t)$ is stationary ergodic in t. Now

$$P(Y(a,t)=Y^*(a,t) \text{ for all sufficiently large } t)=1 \quad \text{for each } a,$$

and hence (iii) holds for $Y(a)=Y^*(a,0)$, and thus (switching negative for positive time) for

$$Y(a)=\max_{t\geq 0}(\mathrm{at}+\sqrt{2}B(t)),\quad a<0. \tag{3.6}$$

Parts (a) and (b) of (iv) are now easy, and (c) is (3.2). Finally, for a general initial distribution $(Y(a,0);\ a<0)$ we can construct some (non-Brownian) $B(t)$, $t<0$, so that (3.5) yields $Y^*(a,0)=Y(a,0)$, and then (3.5) defines the exponential process with this initial distribution.

For the parking lot process $S(m, t)$, the exponential process describes the heavy-traffic limit for the distribution of empty spaces in regions between $\lambda(1-\epsilon)$ and $\lambda - 0(\lambda^{1/2})$. To say this precisely, fix $\frac{1}{2} < b < 1$. Define

$$Y_\lambda(a, t) = \lambda^{b-1}\{(\lambda + a\lambda^b) - S(\lambda + a\lambda^b, \lambda^{1-2b}t)\}; \quad a < 0. \tag{3.7}$$

This means we first slow down time by a factor λ^{2b-1}, so that cars arrive at rate λ^{2-2b} and each car departs at rate λ^{1-2b}. Then we count the number of empty spaces amongst spaces 1 through $\lambda + a\lambda^b$; it turns out this has order λ^{1-b}, so we normalize by that factor. In the following limit theorem, suppose the parking lot process and the exponential process are started in their stationary distributions.

Proposition 3.8. *As $\lambda \to \infty$ the space-time process $(Y_\lambda(a, t);\ a < 0,\ t \geq 0)$ converges in distribution to the exponential process $(Y(a, t);\ a < 0,\ t \geq 0)$. In particular, the stationary distribution $(Y_\lambda(a);\ a < 0)$ converges to the stationary distribution $(Y(a);\ a < 0)$ of the exponential process.*

Technical note. For fixed t the spatial process $Y_\lambda(a, t)$ is considered as a random element of $\mathscr{D} = D(-\infty, 0)$. The space-time process is then considered as a random element of $D([0, \infty), \mathscr{D})$.

Sketch of proof. Fix a_1. Convergence of the stationary marginal

(a) $Y_\lambda(a_1) \xrightarrow{\mathscr{D}} Y(a_1)$ as $\lambda \to \infty$

can be deduced from (1.1) via calculus. We now want to prove convergence of the 1-dimensional processes

(b) $(Y_\lambda(a_1, t);\ t \geq 0) \xrightarrow{\mathscr{D}} (Y(a_1, t);\ t \geq 0)$.

This can be done by appealing to standard results [1, Section 1.9] about weak convergence of discrete-space 1-dimensional Markov processes to diffusions. The essential condition to be verified is that, for small $\delta > 0$, the increment $\Delta Y_\lambda(a_1, t) = Y_\lambda(a_1, t+\delta) - Y_\lambda(a_1, t)$ satisfies

$$E(\Delta Y_\lambda(a_1, t) | Y_\lambda(a_1, t)) \approx a_1\delta + o(\delta),$$

$$\mathrm{var}(\Delta Y_\lambda(a_1, t) | Y_\lambda(a_1, t)) \approx 2\delta + o(\delta)$$

as $\lambda \to \infty$. Because of the rescaling (3.7), this is equivalent to showing that the increment $\Delta S(m, t) = S(m, t+\hat{\delta}) - S(m, t)$ (where $m = \lambda + [a_1\lambda^b]$ and $\hat{\delta} = \lambda^{1-2b}\delta$) satisfies

(c) $E(\Delta S(m, t) | S(m, t)) \approx -\lambda^b\{a_1\hat{\delta} + o(\hat{\delta})\}$,

(d) $\mathrm{var}(\Delta S(m, t) | S(m, t)) \approx \lambda\{2\hat{\delta} + o(\hat{\delta})\}$.

Condition on $S(m, t) = m - s^*$, say. The numbers of arrivals and departures of cars in spaces 1 through m in a time interval $\hat{\delta}$ will be approximately independent Poisson variables with means

(e) $\lambda\hat{\delta}$ (arrivals), $\quad (m - s^*)\hat{\delta} = (\lambda + a_1\lambda^b - s^*)\hat{\delta}$ (departures).

The conditional mean (resp. variance) of $\Delta S(m, t)$ is therefore approximately the difference (resp. sum) of the means in (e); this verifies (c) and (d), since $s^* = 0(\lambda^{1-b}) = o(\lambda^b)$ by (a).

Now let $a_1 < a_2$, and consider convergence of the 2-dimensional processes

(f) $((Y_\lambda(a_1, t), Y_\lambda(a_2, t)); t \geq 0) \xrightarrow{\mathscr{D}} ((Y(a_1, t), Y(a_2, t)); t \geq 0)$.

Let $m_i = \lambda + [a_i\lambda^b]$. Arguing as above,

$$\operatorname{var}(\Delta S(m_2, t) - \Delta S(m_1, t) \mid S(\cdot, t)) \approx (a_2 - a_1)\lambda^b$$

provided spaces 1 through m_1 do not all become occupied. Rescaling, we get

(g) $\operatorname{var}(\Delta Y_\lambda(a_2, 2) - \Delta Y_\lambda(a_1, t) | Y_\lambda(\cdot, t)) \to 0$ as $\lambda \to \infty$

for $Y(a_1, t)$ bounded away from 0. We now prove (f) as follows. By (b) the family of 2-dimensional process in (f) is tight, and also by (b) any subsequential limit process has marginal processes of the form (3.1) for possibly-different Brownian motions $B_i(t)$. But (g) forces these two Brownian motions to be the same. Thus it suffices to show that the 2-dimensional diffusion $(Y(a_1, t), Y(a_2, t))$ is *uniquely* determined by the requirement that the marginals satisfy (3.1) with the same $B(t)$; and this is true because in (3.1) we can write each of $Y(a, t)$ and $B(t)$ as a path-to-path function of the other, using (3.4) one way and

$$X(a, t) = Y(a, t) - L(a, t), \quad L \text{ local time of } Y \text{ at } 0,$$

the other way.

Similarly we get convergence of finite-dimensional processes in (f). Finally, getting function-space tightness (in the space variable a) presents no difficulty since $Y_\lambda(a, t)$ is increasing in a and $Y(a, t)$ is increasing continuous in a.

Remark. Since the geometric process and the exponential process occur as limits for adjacent regions of the parking lot, and one would expect there to be some kind of "compatibility" condition between the two processes. It turns out, by copying the argument above, that the exponential process is a limit of rescaled geometric processes.

Proposition 3.9. *Let $(N(x, t); -\infty < x < 1, t \geq 0)$ be the stationary space-time geometric process, extended to $x < 0$ by putting $N(x, t) = 0$ for $x < 0$. As $K \to \infty$ the processes $(K^{-1}N(1 + a/K, K^2t); a < 0, t \geq 0)$ converge in distribution to the stationary space-time exponential process $Y(a, t)$. In particular, the stationary distributions $(K^{-1}N(1 + a/K); a < 0)$ converge to the stationary distribution $(Y(a); a < 0)$.*

From the viewpoint of heavy-traffic limit theory for the parking lot process, the exponential process and Proposition 3.8 are of limited interest, since the natural questions about the parking lot process do not concern this space-range. On the other hand, the exponential process seems of interest in itself as a simple example of coupled diffusions. Using Proposition 3.9 we can obtain some distributional results about $(Y(a); a < 0)$ by taking limits in corresponding results (2.15–2.18) for the geometric process, as follows.

Corollary 3.10. *Let $a<b<0$.*

(i) $E(Y(b)-Y(a)\mid Y(a)=y)=(1-b/a)(y+a^{-1}-b^{-1})$.

(ii) $E(Y(b)-Y(a))Y(a)=(b/a)(b^{-2}-a^{-2})$.

(iii) $E(Y(b)-Y(a)\mid H_y=a)=(1-b/a)(y-a^{-1}-b^{-1})$, *where* $H_y=\min\{a\colon\ Y(a)=y\}$, $y>0$.

(iv) *The joint Laplace transform*

$$G(\theta,\mu)=E\exp(-\theta Y(a)-\mu(Y(b)-Y(a)))$$

is given by

$$G(\theta,\mu)=\frac{-b(\mu+(\theta-\mu)/Q)}{\theta(\theta+a)-\mu(b-a)};\ Q=1-\tfrac{1}{2}\mu^{-1}(a+\sqrt{a^2+4(b-a)\mu}).$$

These results could also be derived directly by using excursion theory, a program being studied in Fresnedo [4]. Fix $a<b<0$; let $\tau(t)$ be inverse local time of $Y(a,t)$ at the reflecting boundary 0; and consider the process $Y(b,\tau(t))$. This is "the process $Y(b,t)$ looked at only when $Y(a,t)=0$" and is a downward skip-free process which can be analyzed using the excursion structure of $Y(a,t)$. This approach, though technically more sophisticated, ought to be computationally easier, and it seems likely that more information could be obtained.

Other natural questions concern the path-regularity of the stationary spatial process $(Y(a);\ a<0)$.

Proposition 3.11. *There exists a positive discontinuous increasing process* $(Q(a);\ -\infty<a<0)$ *such that*

$$\frac{\mathrm{d}Y}{\mathrm{d}a}(a)=Q(a)\quad \text{a.s. for each fixed } a.$$

Proof. Let $Y(a,t)$ be the stationary exponential process, extended to $-\infty<t<\infty$. Define

$$Q(a)=\min\{t\geq 0\colon\ Y(a,-t)=0\}\quad (\text{note "}-t\text{"}). \tag{3.12}$$

Then $Q(a)$ is increasing and discontinuous. We will show

(a) $Y(a+\varepsilon)\leq Y(a)+\varepsilon Q(a+\varepsilon)$, $\varepsilon>0$.

(b) $(\mathrm{d}/\mathrm{d}a)EY(a)=EQ(a)$,

and then straightforward analysis establishes the result.

To prove (a), observe, from (3.1),

$$\mathrm{d}(Y(a+\varepsilon,t)-Y(a,t))\leq\varepsilon\,\mathrm{d}t \text{ on any interval } (t_1,t_2)$$

$$\text{where } Y(a+\varepsilon,t)>0.$$

Applying this to the interval $(-Q(a+\varepsilon), 0)$ gives (a). To prove (b), observe that for a fixed a the process $Y(a, t)$ is time-reversible. So

$$\begin{aligned} E(Q(a) \mid Y(a,0)=y) &= \text{mean first hitting time for} \\ &\quad \text{Brownian motion with drift } a \text{ from } y \text{ to } 0 \\ &= -y/a, \end{aligned}$$

and hence $EQ(a) = -a^{-1}EY(a)$. Since $EY(a) = -a^{-1}$, we obtain (b).

Remark. The process $(Y(a))$ is not Markov, which partly explains a paradoxical property of Corollary 3.10: that the quantities in (i) and (iii) are unequal even though the conditioning events $\{Y(a)=y\}$ and $\{H_y = a\}$ are essentially identical. The process $(Q(a))$ is (non-homogeneous) Markov, and can again be studied via excursion theory [4].

4. The truncated Normal process

Let $Z(\infty, t)$ be a Ornstein–Uhlenbeck process, that is a diffusion on $(-\infty, \infty)$ with drift $\mu(z) = -z$ and variance 2. In stochastic differential equations language,

$$\mathrm{d}Z(\infty, t) = -Z(\infty, t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B(t). \tag{4.1}$$

This process has stationary distribution $Z(\infty) = \text{Normal}(0,1)$. For fixed a, $-\infty < a < \infty$, let $Z(a, t)$ be the Ornstein–Uhlenbeck process restricted to $(-\infty, a]$ by a reflecting boundary at a:

(i) $\mathrm{d}Z(a, t) = -Z(a, t)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B(t)$ on $\{Z(a, t) < a\}$ (4.2)

(ii) $Z(a, t)$ is reflecting at a.

This has stationary distribution $Z(a)$ which is Normal(0, 1) restricted to $(-\infty, a]$. As in Section 3, we now construct a space-time process $Z(a, t)$ by varying a: we call this the *truncated Normal process.*

For $f(a)$, $-\infty < a \leq \infty$, consider the condition

$$f(a) \text{ is increasing, finite, and } f(b) - f(a) \leq b - a \text{ for } b > a \tag{4.3}$$

This of course implies f is continuous.

Proposition 4.4. *Let* $(Z(a, 0);\ -\infty < a \leq \infty)$ *have paths satisfying* (4.3), *independent of Brownian motion* $B(t)$. *Then there exists a process* $(Z(a, t);\ -\infty < a \leq \infty, 0 \leq t < \infty)$ *such that*

(i) $Z(\infty, t)$ *is the unrestricted Ornstein–Uhlenbeck process* (4.1);

(ii) *for each* $-\infty < a < \infty$, *the process* $Z(a, t)$ *is the Ornstein–Uhlenbeck process on* $(-\infty, a]$ *with a reflecting boundary at a, as at* (4.2);

(iii) *for each t the sample paths* $Z(a, t)$ *satisfy* (4.3);

(iv) *as* $t \to \infty$ *the processes* $(Z(a, t);\ -\infty < a \leq \infty)$ *converge in distribution to a stationary distribution* $(Z(a);\ -\infty < a \leq \infty)$;

(v) *the stationary distribution* $(Z(a);\ -\infty < a \leq \infty)$ *satisfies* (4.3) *and*

(a) $\lim_{a\to-\infty} Z(a) = -\infty$; $\lim_{a\to\infty} Z(a) = Z(\infty)$;

(b) *$Z(a)$ has the Normal* $(0, 1)$ *distribution restricted to* $(-\infty, a)$.

Remarks. (a) In contrast with the exponential process of Section 3, there is no simple explicit representation for $Z(a, t)$.

(b) The path properties (4.3) are natural for the heavy-traffic limit interpretation below.

Proof. For fixed a, the stochastic differential equation (4.2) has a strong solution [5, Section 23], that is to say a solution where $Z(a, t)$ is a function of $Z(a, 0)$ and $(B(u); 0 \leq u \leq t)$. Given such a solution $Z(a, t)$ for each rational a (to avoid worrying about sets of probability zero), let us consider the path properties of $Z(a, t)$ as a varies. Fix rational $a < b$. Let

$$D(t) = Z(b, t) - Z(a, t).$$

From the definition (4.2),

$$\begin{aligned}
&\text{(a)} \quad \mathrm{d}D(t) = -(b-a)D(t)\,\mathrm{d}t \text{ on } \{Z(a, t) < a,\ Z(b, t) < b\},\\
&\text{(b)} \quad \phantom{\mathrm{d}D(t)} \geq -(b-a)D(t)\,\mathrm{d}t \text{ on } \{Z(b, t) < b\},\\
&\text{(c)} \quad \phantom{\mathrm{d}D(t)} \leq -(b-a)D(t)\,\mathrm{d}t \text{ on } \{Z(a, t) < a\}.
\end{aligned} \tag{4.5}$$

Given $D(0) > 0$, we will show that $D(t) > 0$ for all $t \geq 0$. For if not, there is some first t_0 where $D(t_0) = 0$. It cannot happen that $Z(b, t_0) = b$, for $Z(a, t_0) \leq a$. But by (b) it cannot happen that $Z(b, t_0) < b$, since there $D(t)$ is decreasing at most exponentially fast.

Now consider

$$C(t) = (Z(a, t) - a) - (Z(b, t) - b).$$

Given $C(0) \geq 0$, we will show that $C(t) \geq 0$ for all $t \geq 0$. At times t_0 when $Z(a, t_0) = a$, we have $Z(b, t_0) \leq b$ and so $C(t_0) \geq 0$. Thus it suffices to show that $C(t)$ cannot become negative during an excursion of $Z(a, t)$ from the boundary a. But on such an excursion,

$$\mathrm{d}C(t) = -\mathrm{d}D(t) \geq (b-a)D(t) \geq 0 \quad \text{by } (c),$$

so $C(t)$ can indeed never become negative.

The positivity of $C(t)$ and $D(t)$ imply that for fixed t, the paths $Z(a, t)$ satisfy (4.3) for rational a. We can now extend to general a by continuity, and verify that the marginal processes $Z(a, t)$ (a fixed) are indeed reflecting Ornstein–Uhlenbeck processes. This completes the proof of (i)–(iii).

For fixed a, one-dimensional theory says $Z(a, t)$ converges in distribution as $t \to \infty$ to a stationary distribution $Z(a)$. The path properties (4.3) imply that as $t \to \infty$ the distributions $(Z(a, t);\ -\infty < a \leq \infty)$ are tight on $D(-\infty, \infty]$. A coupling argument, as in the proof of Proposition 2.2, completes the proof of (iv). And (v) is straightforward.

For the parking lot process $S(m, t)$, the truncated Normal process describes the heavy-traffic limit for the distribution of empty spaces in the region $\lambda \pm 0(\lambda^{1/2})$, or for the "overflow" past m (that is, the number of cars parked to the right of m) for m in this range. To say this precisely, for the parking lot process with arrival rate λ write

$$Z_\lambda(a, t) = \lambda^{-1/2}\{S(\lambda + a\lambda^{1/2}, t) - \lambda\}, \quad -\infty < a \leq \infty, t \geq 0. \tag{4.6}$$

To understand this, recall that $S(\infty, t)$, the total number of cars parked, evolves as the $M/M/\infty$ queue which has a Poisson(λ) $\approx$ Normal(λ, λ) stationary distribution. So $Z_\lambda(\infty, t)$ is the natural standardization whose stationary distribution is approximately Normal(0, 1), and it is well known [8] that the standardized process $Z_\lambda(\infty, t)$ approximates the Ornstein-Uhlenbeck process $Z(\infty, t)$ (note there is no rescaling of time here). Here is the limit theorem, for processes started in their stationary distribution.

Proposition 4.7. *As $\lambda \to \infty$ the space-time processes $(Z_\lambda(a, t);\ -\infty < a \leq \infty;\ t \geq 0)$ converge in distribution to the truncated Normal process $(Z(a, t);\ -\infty < a \leq \infty;\ t \geq 0)$. In particular, the stationary processes $(Z_\lambda(a);\ -\infty < a \leq \infty)$ converge to the stationary truncated Normal process $(Z(a);\ -\infty < a \leq \infty)$.*

This result is described, in less formal language, in [13, Section 6]. The details of the proof are similar to those of Proposition 3.8. The asymptotic overflow distribution $Z(\infty) - Z(a)$, and non-asymptotic corrections for $S(\infty) - S(m)$, and the corresponding joint distributions $(Z(a), Z(\infty) - Z(a))$, are discussed in detail in [13, Sections 4 and 5], and we will not restate them here.

Let us just state the analog of Proposition 3.11.

Proposition 4.8. *There exists a positive increasing discontinuous process $Q(a)$ such that $(d/da)Z(a) = \exp(-Q(a))$ a.s. for each a.*

In fact, this holds for

$$Q(a) = \min\{t > 0: Z(a, -t) = a\} \tag{4.9}$$

where $Z(a, t)$ is the stationary space-time truncated Normal process, extended to $-\infty < t < \infty$.

5. The extremal process

For the stationary storage process $S(m, t)$ with arrival rate λ, let $R(t)$ be the position of the rightmost occupied space at time t. Coffman et al. [2] use ingenious generating function arguments to obtain an explicit formula for the stationary distribution R:

$$P(R>m)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left\{\sum_{k=0}^{m}\binom{m}{k}\frac{(n+k)!}{\lambda^{n+k+1}}\right\}^{-1},\quad m\geqslant 0. \tag{5.1}$$

We shall use different methods to obtain the heavy-traffic limit of R.

Let N be the Poisson point process on $\mathbb{R}^2$ with intensity ν given by

$$\nu(\mathrm{d}t,\mathrm{d}x)=\mathrm{e}^{-x}\,\mathrm{d}t\,\mathrm{d}x.$$

Define

$$\xi(t)=\max\{x\colon (t',x)\in\mathcal{N} \text{ for some } t'\in(t-1,t]\}.$$

Then $(\xi(t))$ is a stationary process, whose marginal distribution ξ is

$$\begin{aligned}P(\xi\leqslant x)&=P(\text{no point of }\mathcal{N}\text{ in }(t-1,t]\times(x,\infty))\\&=\exp(-\nu(t-1,t]\times(x,\infty))=\exp(-\mathrm{e}^{-x}).\end{aligned} \tag{5.2}$$

Thus ξ has the classical "double exponential" extreme value distribution, and $\xi(t)$ can be regarded as a stationary "extremal process" (definitions of that term vary). Now define $a=a(\lambda)$ by

$$a\phi(a)\log\lambda^{1/2}=1 \tag{5.3}$$

where ϕ is the standard Normal density. The approximation

$$a^2(\lambda)\approx 2\{\log\log\lambda^{1/2}+\log\{2\log\log\lambda^{1/2}\}^{1/2}\} \tag{5.4}$$

may be substituted in the sequel. Define the normalized process

$$\xi_\lambda(t)=\lambda^{-1/2}a(\lambda)\{R(t\log\lambda^{1/2})-\lambda-a(\lambda)\lambda^{1/2}\}. \tag{5.5}$$

Proposition 5.6. *As $\lambda\to\infty$ the processes $(\xi_\lambda(t);\ t\geqslant 0)$ converge, in the sense of finite-dimensional distributions, to $(\xi(t);\ t\geqslant 0)$. In particular the stationary distributions ξ_λ converge to ξ.*

Remarks. (a) More crudely, for large λ,

$$R\approx\lambda+\sqrt{2\lambda\log\log\lambda^{1/2}}. \tag{5.7}$$

This sharpens the bound $R\leqslant\lambda+c\sqrt{\lambda\log\lambda}$ given in [2]. Note that R has slightly *less* chance variability than does $S(\infty)$, order $a^{-1}\lambda^{1/2}$ instead of $\lambda^{1/2}$.

(b) It is not clear how to derive this limit from the analytic expression (5.1).

(c) In Proposition 5.6 we do not have convergence in the usual topology on $D[0,\infty)$, because one jump of $\xi(t)$ may approximate several nearby smaller jumps of $\xi_\lambda(t)$.

(d) We consider these stationary processes extended to $-\infty<t<\infty$.

Let $S(t)=S(\infty, t)$ be the total number of parked cars; recall this process evolves as the M/M/∞ queue. Let $b=b(\lambda)=\log\lambda^{1/2}$, and let $a=a(\lambda)$ be as at (5.3). Let

$$M_\lambda(t)=\lambda^{-1/2}a\{S(t/b)-\lambda-a\lambda^{1/2}\}$$

$$M_\lambda^h(t)=\sup_{t-h\leq u\leq t} M_\lambda(u)$$

and let

$$\xi^h(t)=\max\{x\colon (t',x)\in\mathcal{N} \text{ for some } t'\in(t-h,t]\}.$$

Proposition 5.6 splits into two parts, as follows.

Proposition 5.8. *Fix $h>0$. As $\lambda\to\infty$ the processes $(M_\lambda^h(t);\ t\geq 0)$ converge, in the sense of finite-dimensional distributions, to $(\xi^h(t);\ t\geq 0)$.*

Proposition 5.9. *$M_\lambda^1(0)-\xi_\lambda(0)\to 0$ in probability as $\lambda\to\infty$.*

Note that Proposition 5.8 involves only the M/M/∞ queue $S(t)$, and no further properties of the parking lot process. Proposition 5.8 is closely related to standard results about extreme values of stationary processes; we will sketch the proof later.

Proof of Proposition 5.9. Let $q(K,t)$ be the chance that, out of a set of K cars parked at time t_0, at least one of these cars is still parked at time t_0+t. Then $q(K,t)=1-(1-\mathrm{e}^{-t})^K$, and so for fixed $\varepsilon>0$ we obtain:

$$\text{if}\quad K/\lambda^{1/2+\varepsilon}\to 0\quad\text{then}\quad q(K,(1+3\varepsilon)b)\to 0, \tag{5.10}$$

$$\text{if}\quad K/\lambda^{1/2-\varepsilon}\to \infty\quad\text{then}\quad q(K,(1-3\varepsilon)b)\to 1. \tag{5.11}$$

Now fix a time t_0, a parking space m_λ and $\varepsilon>0$. Suppose $S(t_0)\geq m_\lambda$. Consider the set of cars parked to the right of space $m_\lambda-\lambda^{1/2-1/2}\varepsilon$. At time t_0 there are at least $\lambda^{1/2-\varepsilon/2}$ such cars; so by (5.11) at time $t_0+(1-3\varepsilon)b$ the chance that at least one such car remains tends to 1 as $\lambda\to\infty$. Applying this fact to the stopping time

$$T=\min_{t\geq-(1-3\varepsilon)b}\{t\colon S(t)\geq m_\lambda\}$$

in place of t_0, we see

$$P\Big(\max_{-(1-3\varepsilon)b\leq t\leq 0} S(t)\geq m_\lambda,\ R(0)<m_\lambda-\lambda^{1/2-\varepsilon/2}\Big)\to 0\quad\text{as }\lambda\to\infty.$$

Rescaling, we see that, for any x,

$$P\Big(\max_{-(1-3\varepsilon)\leq t\leq 0} M_\lambda(t)\geq x,\ \xi_\lambda(0)<x-a\lambda^{-\varepsilon/2}\Big)\to 0\quad\text{as }\lambda\to\infty.$$

Thus for fixed $\delta>0$ we conclude

$$P(M_\lambda^{1-3\varepsilon}(0)>\xi_\lambda(0)+\delta)\to 0\quad\text{as }\lambda\to\infty. \tag{5.12}$$

This gives a bound in one direction. For the other, fix $\varepsilon > 0$. A result from [2], which can also be derived from (1.3) by calculus, is

$$P(R(0) \leq \lambda + \lambda^{1/2+\varepsilon/2}) \to 1 \quad \text{as } \lambda \to \infty. \tag{5.13}$$

Also $R(0) \geq S(0)$, and using Proposition 5.8 we get

$$R(\lambda < R(0) \leq \lambda + \lambda^{1/2+\varepsilon/2}) \to 1 \quad \text{as } \lambda \to \infty. \tag{5.14}$$

We want to show

$$P(R(0) > \max_{-(1+3\varepsilon)b \leq t \leq 0} S(t)) \to 0 \quad \text{as } \lambda \to \infty. \tag{5.15}$$

For this event can only happen if the rightmost car at time 0 (in space r, say) has remained there since time $-(1+3\varepsilon)b$. By (5.14) we may suppose $r \in (\lambda, \lambda + \lambda^{1/2+\varepsilon/2})$; but then by (5.10) the chance that any car from this range remains from time $-(1+3\varepsilon)b$ until time 0 is asymptotically 0. This establishes (5.15). Rescaling gives

$$P(M_\lambda^{1+3\varepsilon}(0) < \xi_\lambda(0)) \to 0 \quad \text{as } \lambda \to \infty. \tag{5.16}$$

It is now straightforward to derive Proposition 5.9 from (5.12), (5.16) and Proposition 5.8.

Sketch of proof of Proposition 5.8. We wish to describe some known results from a different viewpoint. Let $X(t)$ be a stationary process. Take z large so that $P(X(0) > z)$ is small, and let $g(z) < z$. If at time t_0 we have $X(t_0) \geq z$ then there is a last time $s < t_0$ and a first time $s' > t_0$ that $X(s) \leq g(z)$; let $\hat{x}$ be the maximum of X over the interval (s, s') and let $\hat{t}$ be the time at which the maximum is attained. By varying t_0 we can construct a time-space point process $\mathcal{N}_z$ whose points $(\hat{t}, \hat{x})$ are the times and heights of "semi-local maxima" of X. Provided $g(z) \to \infty$ and $P(X(0) > z)/P(X(0) > g(z)) \to 0$ as $z \to \infty$, then under weak regularity conditions the process $\mathcal{N}_z$ for z large will be essentially independent of the precise choice of $g(z)$, and we can regard the $\mathcal{N}_z$ as the restriction to $\{(t, x): x \geq z\}$ of a single point process $\mathcal{N}_X$. For $\alpha, \beta, \gamma > 0$ the rescaling map

$$(t, x) \to (t/\beta, (x-\alpha)/\gamma)$$

maps a point process $\mathcal{N}$ to another point process $r_{\alpha,\beta,\gamma}(\mathcal{N})$ say. Thus one can look for asymptotic results of the form

$$r_{\alpha_n,\beta_n,\gamma_n}(\mathcal{N}_X) \xrightarrow{\mathcal{D}} \mathcal{N} \quad \text{for constants } \alpha_n, \beta_n, \gamma_n$$

and such results describe the times and heights of high-level semi-local maxima of the stationary process X. For the Ornstein–Uhlenbeck process $Z(t) = Z(\infty, t)$ of Section 4, the point process $\mathcal{N}_Z$ is, for x large, approximately the Poisson point process of intensity

$$\nu(\mathrm{d}t, \mathrm{d}x) = x^2\phi(x)\,\mathrm{d}t\,\mathrm{d}x, \tag{5.17}$$

where ϕ is the standard Normal density. To say this as a limit theorem, use the equations

$$\beta\alpha\phi(\alpha)=1 \qquad \alpha\gamma=1 \tag{5.18}$$

to define α, γ as functions of β for large β. These equations are chosen so that, for the Poisson point process $\hat{\mathcal{N}}$ with intensity (5.17),

$$r_{\alpha,\beta,\gamma}(\hat{\mathcal{N}}) \xrightarrow{\mathcal{D}} \mathcal{N} \quad \text{as } \beta\to\infty$$

where $\mathcal{N}$ is the Poisson point process of intensity e^{-x}. The precise result for the Ornstein-Uhlenbeck process is

Theorem 5.19. *$r_{\alpha,\beta,\gamma}(\mathcal{N}_Z) \xrightarrow{\mathcal{D}} \mathcal{N}$ as $\beta\to\infty$.*

Results of this type are discussed in [11, Section 9.5, 12.4], using the notion of "ε-upcrossing" in place of "semi-local maximum".

Now let $S(t)$ be the $M/M/\infty$ queue and as in Section 4 normalize to

$$Z_\lambda(t)=\lambda^{-1/2}(S(t)-\lambda).$$

Let $\mathcal{N}_\lambda$ be the point process of semi-local maxima derived from Z_λ. Since $(Z_\lambda(t);\ t\geq 0)$ converges to $(Z(t);\ t\geq 0)$ we have $\mathcal{N}_\lambda \xrightarrow{\mathcal{D}} \mathcal{N}_Z$ and so

$$r_{\alpha,\beta,\gamma}(\mathcal{N}_\lambda) \xrightarrow{\mathcal{D}} r_{\alpha,\beta,\gamma}(\mathcal{N}_Z) \text{ as } \lambda\to\infty, \quad \text{fixed } \alpha, \beta, \gamma. \tag{5.20}$$

Results (5.19) and (5.20) are essentially known; the essence of Proposition 5.8 is that we can combine the limiting operations as follows.

Proposition 5.21. *Let $\beta=\log\lambda^{1/2}$ and define α, γ by* (5.18). *Then $r_{\alpha,\beta,\gamma}(\mathcal{N}_\lambda) \xrightarrow{\mathcal{D}} \mathcal{N}$ as $\lambda\to\infty$.*

Our arguments so far were designed to convince the reader that this result is natural; to actually *prove* Proposition 5.21 it seems necessary to start from the beginning and modify the arguments of [11] to apply to the $M/M/\infty$ queues $S(t)$, an undertaking we shall omit.

Finally, the process $M_\lambda(t)$ occurring in Proposition 5.8 was defined so that its point process of semi-local maxima is $r_{\alpha,\beta,\gamma}(\mathcal{N}_\lambda)$, which by Proposition 5.21 approximates $\mathcal{N}$. The processes M_λ^h and ξ^h are constructed from these point processes in the same way, that is by taking the highest-level point which occurs in the previous h time units, and so convergence of these constructed processes follows fairly easily from convergence of the underlying point processes.

6. Miscellany

(A) *Function-valued diffusions.* The exponential process and the truncated Normal process are simple examples of diffusions with values in a space of continuous functions. Such diffusions (under alternative names like "stochastic functional differential equations" or "measure-valued diffusion") arise in several contexts [3, 9, 12, 15], and will perhaps form a standard subfield of Markov process theory in the near future. Different examples suggest different theoretical questions; our examples suggest the following problem.

Problem 6.1. Can one describe explicitly all function-valued diffusions $X(a, t)$ with the property that for each d and $(a_1, \dots, a_d)$, the process $(X(a_1, t), \dots, X(a_d, t))$ is a d-dimensional diffusion, and such that there is a limiting stationary distribution $(X(a))$ with continuous paths?

(B) *Approximate independence of subprocesses.* Let $(X_\lambda(t);\ t \geq 0)$ be a stationary process, depending on a parameter λ, taking values in a space S. For $i = 1, 2$ let $X^i_\lambda(t) = f_i(\lambda, X_\lambda(t))$ for some functions f_i taking values in a space S_i. Think of $X_\lambda(t)$ as some complicated process, and the $X^i_\lambda(t)$ as simpler subprocesses defined by ignoring much of X_λ. A principle well known in physics applications ("slaving", in the terminology of [6]) states: if the times taken for the two subprocesses $X^i_\lambda(t)$ to relax to equilibrium are of different orders of magnitude, then at equilibrium the subprocesses $X^i_\lambda(t)$ are approximately independent. It is straightforward to formulate and prove this as a limit theorem.

Proposition 6.2. *Suppose for $i = 1, 2$ there are constants $c_i(\lambda)$ and processes $Z^i(t)$ in $D([0, \infty), S_i)$ such that*

(i) $X^i_\lambda(tc_i(\lambda)) \xrightarrow{\mathcal{D}} Z^i(t)$ *as $\lambda \to \infty$, in the sense of finite-dimensional distributions;*

(ii) $c_1(\lambda)/c_2(\lambda) \to 0$ *as $\lambda \to \infty$;*

(iii) $Z^2(t)$ *is ergodic.*

Then, as $\lambda \to \infty$,

(iv) $(X^1_\lambda(tc_1(\lambda)), X^2_\lambda(tc_2(\lambda))) \xrightarrow{\mathcal{D}} (Z^1(t), Z^2(t))$ *in the sense of finite-dimensional distributions, and in particular*

(v) $(X^1_\lambda(0), X^2_\lambda(0)) \xrightarrow{\mathcal{D}} (Z^1(0), Z^2(0))$,

where in (iv) *and* (v) *the limit processes Z^i are independent.*

Our parking lot process offers a very nice application of this principle. In Propositions 2.5, 3.8, 4.7 and 5.6 we have seen that as $\lambda \to \infty$ the subprocesses $M_\lambda(x, t)$, $Y_\lambda(a, t)$, $Z_\lambda(a, t)$ and $\xi_\lambda(t)$ converge to the limit processes $N(x, t)$, $Y(a, t)$, $Z(a, t)$ and $\xi(t)$. Each result involves a different time rescaling (by factors λ^{-1}, λ^{1-2b}, 1, $\log \lambda^{1/2}$). Proposition 6.2 shows that for the stationary parking lot process, these subprocesses converge jointly to the limit $((N(x)); (Y(a)); (Z(a)); \xi)$ in which the four limit subprocesses are independent. Informally, for large λ the stationary behavior of the parking lot process is approximately independent on the four regions of the parking lot. In particular, for large λ the position R of the rightmost car is appoximately independent of the number $S(\infty)$ of parked cars, a result which is perhaps surprising until one realizes that $R(t)$ and $S(\infty, t)$ evolve on different time-scales.

(C) *Relations between limit processes, and self-similarity.* From partial sums of i.i.d. Bernoulli variables we can derive, by normalizing and taking limits, two types of limit process: Poisson process and Brownian motion. Moreover from the Poisson process we can derive, by again normalizing and taking limits, Brownian motion. Our parking lot process exhibits similar but more complex behavior. We have three limit space-time processes. Proposition 3.9 says that from the right tail of the geometric process we can derive the exponential process. Similarly, from the left tail of the truncated Normal process we can also derive the exponential process.

Proposition 6.3. *Let* $(Y(a))$ *and* $(Z(a))$ *be the stationary distributions of the exponential and truncated Normal processes. Let* $\bar{Z}(a) = a - Z(a)$. *Then* $(K\bar{Z}(aK);\ -\infty < a < 0) \xrightarrow{\mathcal{D}} (Y(a);\ -\infty < a < 0)$ *as* $K \to \infty$.

Just as we cannot recover the Poisson process from Brownian motion, we cannot recover the other processes from the exponential process. One explanation is that, like Brownian motion, the exponential process has a self-similarity property that the other processes do not have (and of course self-similarity is preserved under normalization and weak limits).

Proposition 6.4. *For* $c > 0$, $(cY(ac);\ a < 0) \overset{\mathcal{D}}{=} (Y(a);\ a < 0)$.

The space-time exponential process thus has the properties of being a stationary diffusion in the time variable, and self-similar in the space variable. Perhaps there is an interesting class of processes with such properties?

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