Combinatorics, Probability and Computing (2005) 00, 000–000.
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

Uniform Multicommodity Flow through the Complete Graph with Random Edge-capacities

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Give random capacities C to the edges of the complete n-vertex graph. Consider the maximum flow Φ_n that can be simultaneously routed between each source-destination pair. We prove that $\Phi_n \to \phi$ where the limit constant ϕ depends on the distribution of C in a simple way, and that asymptotically one need use only one- and two-step routes. The proof uses a reduction to the following random graph problem. Make a random graph in which each edge is blue with probability p_b or scarlet with probability $p_s < p_b/2$ or neither. Then we can find edge-disjoint triangles, with exactly one scarlet edge and exactly two blue edges, which exhaust almost all the scarlet edges. We show this by analyzing a simple greedy algorithm for which one expects a differential equation approximation. The approximation can be justified in a direct way using the local weak convergence methodology, which may more generally be useful in formalizing differential equation approximations in randomized graph algorithm contexts.

1. Introduction

This paper is part of a project studying optimal flows through random networks, where a network has both a graph structure and extra structure such as capacities and costs on edges, and where we are in the "multicommodity flow" setting with simultaneous flows between each source-destination pair. We plan a survey elsewhere. Possible models span a broad spectrum from realistic to mathematically tractable, and at the latter end are models based on the complete graph. Including study of such models within a project is natural both for mathematical completeness and for comparison purposes.

Consider first the setting of an arbitrary finite connected undirected graph G. Let $\phi > 0$. A flow of volume $\phi/2$ between vertex v and vertex w has net out-flow $= \phi/2$ at v, net out-flow $= -\phi/2$ and w, and zero net out-flow at other vertices. For such a flow write $f_{v,w}(e) \geq 0$ for the absolute value of the flow volume across an undirected edge e. Suppose we have such a flow simultaneously for each ordered pair (v, w) with $w \neq v$; call this collection a *uniform flow* of volume ϕ and write $f(e) := \sum_{(v,w)} f_{v,w}(e)$

† Research supported by N.S.F Grant DMS-0203062.

for the combined volume of flow across the undirected edge e. Suppose now we are given capacities $C(e)$ for edges e. Then the *maximum uniform flow volume* (MUFV) is defined to be the largest ϕ such that there exists a uniform flow of volume ϕ which satisfies the capacity constraints

$$f(e) \leq C(e) \ \forall e. \tag{1.1}$$

One modeling paradigm, seeking to combine the spatial inhomogeneity of real networks with mathematical tractability, is to consider some standard family G_n of n-vertex graphs, and to assume the edge-capacities $C(e)$ are random (specifically, are i.i.d. copies of a reference r.v. C). Now the MUFV is a r.v. Φ_n, and one can seek to study its $n \to \infty$ behavior.

Apparently, and somewhat surprisingly, such questions have not been studied before. There is literature [4, 3, 2, 5] on flows with a single source-destination pair and on flows from top to bottom of a square, but these fall into the one-commodity setting of the max-flow min-cut theorem, rather than our multicommodity setting.

In this paper we consider the complete graph, and in [1] we consider a similar problem on the $m \times m$ square grid. An interesting observation is that in both these models the limit constants for Φ_n depend on the distribution of C (not just on its expectation EC), but for rather different reasons in the two models. An intermediate model is the cube $\{0,1\}^d$, and here we conjecture that the limit constant does depend only on EC when C is bounded away from zero.

1.1. Statement of results

Consider the complete n-vertex graph whose edges e have independent random capacities $C(e)$ whose common distribution satisfies

$$P(C \geq c_0) = 1; \quad \text{some } c_0 > 0. \tag{1.2}$$

$$EC < \infty. \tag{1.3}$$

Note that the function

$$\phi \to 2E\max(\phi - C, 0) \ - \ E\max(C - \phi, 0)$$

in continuous and strictly increasing from $-EC$ to ∞ as ϕ increases from 0 to ∞, and so we can define a constant $0 < \phi_* < \infty$ as the unique solution of

$$E\max(C - \phi_*, 0) = 2E\max(\phi_* - C, 0). \tag{1.4}$$

Theorem 1.1. *Under assumptions (1.2,1.3) the MUFV Φ_n satisfies $\Phi_n \to \phi_*$ in probability as $n \to \infty$.*

The intuition is very simple. Suppose we wish to route a uniform flow of volume ϕ. First route as much flow as possible across the direct edge, that is route volume $\min(\phi, C_{v,w})$ across edge $\{v,w\}$. This leaves an unsatisfied demand for volume $\max(\phi - C_{v,w}, 0)$ of flow. Now the mean surplus capacity per edge is $E\max(C - \phi, 0)$. We try to route the unsatisfied demand via 2-step paths with surplus capacity; for this to work it seems evidently necessary that

$$E \max(C - \phi, 0) \geq 2E \max(\phi - C_{v,w}, 0).$$

Conversely, this should be sufficient because the set of edges with surplus capacity forms a dense random graph which should be sufficiently well-connected to permit the desired 2-step paths.

The "necessary" part is indeed easy to formalize (Lemma 2.1). For the converse, we start by proving (section 2) a reduction to the following "random graph" result. To motivate this reduction, consider the case where the edge-capacity C takes only values $\{0, 1, 2\}$ and where we seek to route a uniform flow of volume 1. Then traffic across capacity-0 edges (colored scarlet, say) needs to be routed through two capacity-2 edges (colored blue, say). Colors are mnemonics for *smaller* and *bigger* capacity.

Proposition 1.2. *Fix $0 < p_s < p_b/2$ with $p_s + p_b \leq 1$. Randomly color the edges of the complete n-vertex graph as blue (probability p_b) or scarlet (probability p_s) or neither (probability $1 - p_b - p_s$). Then there exists a collection of edge-disjoint triangles, each triangle having one scarlet edge and two blue edges, such that the number $N_n(v)$ of scarlet edges incident at v which are not is some triangle satisfies*

$$n^{-1} \max_v N_n(v) \to 0 \text{ in probability.} \tag{1.5}$$

Proposition 1.2 is proved in section xxx by analysing the natural greedy algorithm. The particular method of justifying the natural differential equation approximation is perhaps of wider methodological interest, so will be discussed at xxx.

The assumption (1.2) is used only in the proof of Lemma 2.3, and we conjecture that it can be eliminated.

xxx mention old paper by Frank Kelly (neat, but not diretly relevant).

2. The reduction argument

2.1. The upper bound

The upper bound in Theorem 1.1 is provided by

Lemma 2.1. $\lim_n P(\Phi_n \geq \phi) = 0, \quad \phi > \phi_*.$

Proof. Fix a realization of the edge-capacities. Suppose a uniform flow of volume ρ exists. For an edge (v, w)

$$\begin{aligned} \sum_e (f_{v,w}(e) + f_{w,v}(e)) &\geq \rho \text{ if } C(v,w) \geq \rho \\ &\geq C(v,w) + 2(\rho - C(v,w)) \text{ if } C(v,w) \leq \rho \end{aligned}$$

because in the latter case volume of at least $\rho - C(v, w)$ must use at least a 2-step route. Combining the two cases,

$$\sum_e (f_{v,w}(e) + f_{w,v}(e)) \geq \min(\rho, C(v,w)) + 2\max(\rho - C(v,w), 0).$$

Summing over edges $e' = (v, w)$ and using the capacity constraint (1.1),

$$\sum_e C(e) \geq \sum_{e'} \left(\min(\rho, C(e')) + 2\max(\rho - C(e'), 0)\right).$$

Dividing by $\binom{n}{2}$ and recalling we supposed that the uniform flow exists, we have shown

$$Q_n := \frac{1}{\binom{n}{2}} \sum_{e'} C(e') - \frac{1}{\binom{n}{2}} \sum_{e'} \left(\min(\rho, C(e')) + 2\max(\rho - C(e'), 0)\right) \geq 0 \text{ on } \{\Phi_n \geq \phi\}.$$

But as $n \to \infty$ the quantity Q_n converges in probability to

$$q := EC - (E\min(\rho, C) + 2E\max(\rho - C, 0)) = E\max(C - \rho, 0) - 2E\max(\rho - C, 0).$$

If $\rho > \rho_*$ then $q < 0$ and hence we must have $\lim_n P(\Phi_n \geq \phi) = 0$. □

2.2. The reduction

In this section we assume the truth of Proposition 1.2 and show how to deduce the lower bound in Theorem 1.1, stated as Lemma 2.3. Note that the condition $\rho < \rho_*$ is equivalent to

$$r := \frac{E\max(\rho - C, 0)}{E\max(C - \rho, 0)} < \frac{1}{2}. \tag{2.1}$$

Lemma 2.2. *Suppose C is integer-valued and bounded, and suppose ρ is an integer satisfying (2.1). Then, for (v, w) outside some random subset B_n of edges, we can construct flows of volume ρ between v and w such that the capacity constraint (1.1) holds and such that*

$$n^{-1} \max_v |\{e \in B_n : \ e \text{ incident at } v\}| \to 0 \text{ in probability.}$$

Proof. We first construct a finite set $\mathcal{M}$ of "marks" μ and a joint distribution $(C; \xi_1, \xi_2, \ldots, \xi_{|C-\rho|})$ which associates $|C - \rho|$ distinct random marks ξ_j with a realization of C, such that the following two properties hold.

(i) For each mark $\mu \in \mathcal{M}$ there exists exactly one value $i = i_s(\mu) \in \{0, 1, \ldots, \rho - 1\}$ such that $P(C = i, \text{ some } \xi_j = \mu) > 0$ and there exists exactly one value $i = i_b(\mu) \in \{\rho + 1, \rho + 2, \ldots\}$ such that $P(C = i, \text{ some } \xi_j = \mu) > 0$.

(ii) For each mark $\mu \in \mathcal{M}$

$$\frac{P(C = i_s(\mu), \text{ some } \xi_j = \mu)}{P(C = i_b(\mu), \text{ some } \xi_j = \mu)} = r$$

for r at (2.1).

The construction is illustrated in Figure 2.2. Set $p_i = P(C = i)$. Partition a line of length $E\max(\rho - C, 0) = \sum_{i=0}^{\rho-1} (\rho - i) p_i$ into $(\rho - i)$ consecutive intervals of lengths p_i (for $i = 0, 1, \ldots, \rho - 1$). Partition a line of equal length $E\max(\rho - C, 0) = rE\max(C - \rho, 0) = \sum_{i>\rho} r(i - \rho) p_i$ into $(i - \rho)$ consecutive intervals of lengths rp_i (for $i = \rho + 1, \rho_2, \ldots$). Now identify the two lines, and let $\mathcal{M}$ be the set of intervals μ arising as the intersection of some interval in the first partition with some interval in the second partition. Given $C = i$, choose random points uniformly (with respect to length) from each of the $|i - \rho|$

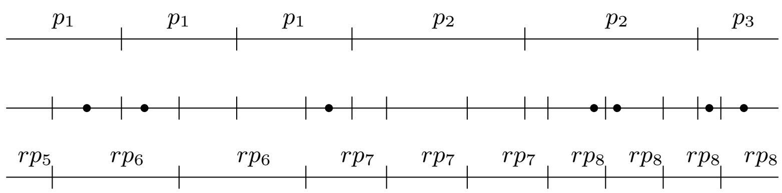

Figure 1. Construction of marks. Here $\rho = 4$. The middle line shows the intervals which are the marks μ. On the left is a typical set of 3 marks associated with a value $C = 1$. On the right is a typical set of 4 marks associated with a value $C = 8$.

intervals of length p_i or rp_i; each such point is in some intersection-interval μ, so let the marks $\xi_1, \ldots, \xi_{|i-\rho|}$ be these intersection-intervals. Properties (i) and (ii) are immediate from the construction.

To prove the lemma, associate each original edge of the complete graph with a realization of $(C; \xi_1, \xi_2, \ldots, \xi_{|C-\rho|})$ and replace the original edge by $|C - \rho|$ edges marked $(\xi_1, \xi_2, \ldots, \xi_{|C-\rho|})$ and colored scarlet (if $C < \rho$) or blue (if $C > \rho$). Now fix a mark μ and consider all edges with mark μ. By (ii) the hypothesis of Proposition 1.2 is satisfied, so we can find edge-disjoint triangles with two scarlet edges and one blue edges, all with mark μ, satisfying (1.5). Repeat for all marks. For each scarlet edge in one of the triangles, route unit flow between its end-vertices by using the two blue edges in the triangle. Then (1.5) and the finiteness of number of marks establish the lemma. □

Lemma 2.3. *Assume (1.2,1.3) and let $\phi < \phi_*$. Then $\lim_n P(\Phi_n \geq \phi) = 1$.*

Proof. Define

$$C_k = \min(2^{-k} \lfloor C2^k - 1 \rfloor, k)$$

for k sufficiently large that $2^{-k} < c_0$ for c_0 at (1.2). So $0 \leq C_k \leq C - 2^{-k}$. Define ρ_k as the largest multiple of 2^{-k} for which

$$E \max(C_k - \rho_k, 0) > 2E \max(\rho_k - C_k, 0).$$

It is easy to check that $\rho_k \uparrow \rho_*$ as $k \to \infty$. Thus it is sufficient to show that, for each fixed large k,

$$P(\text{uniform flow of volume } \rho_k \text{ exists}) \to 0 \text{ as } n \to \infty. \qquad (2.2)$$

By applying Lemma 2.2 to the integer-valued quantities $2^k C_k$ and $2^k \rho_k$, then rescaling, we find that for (v, w) outside some random subset B_n of edges such that

$$q_n := \max_v \tfrac{q_n(v)}{n-2} \to 0 \text{ in probability}$$

$$q_n(v) := |\{e \in B_n : \ e \text{ incident at } v\}|$$

we can construct flows of volume ρ_k between v and w such that the total flow volume $f(e)$ satisfies the capacity constraints $f(e) \leq C_k(e)\ \forall e$. Because $C_k(e) \leq C(e) - 2^{-k}$, each edge has unused capacity of at least 2^{-k}. So between each pair $(v,w) \in B_n$ route a flow of volume ρ_k by spreading the flow uniformly over all 2-step paths. The volume of flow across an edge (v',w') created in this way is at most

$$\rho_k \left(\tfrac{q_n(v')}{n-2} + \tfrac{q_n(w')}{n-2} \right) \leq 2\rho_* q_n.$$

So if $2\rho_* q_n < 2^{-k}$ there exists a feasible flow of volume ρ_k, establishing (2.2). $\square$

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