“Spin Glasses: A Challenge for Mathematicians” by Michel Talagrand (Springer, 2003, hardcover, 586pp., $159.00, ISBN 3-540-00356-8).

Part of my standard advice to graduate students seeking a career in mathematics research is that, sometime during their second or third year, they should spend six months getting to grips with the core technical details of some currently active area of research. This is most conveniently done by reading every line of a recent monograph. I speak from personal experience; reading, 30 years ago, Billingsley's monograph [1] on weak convergence has served me well ever since. The monograph under review is a ideally suited for this purpose, not only by virtue of its content and style being “core technical details of some currently active area of research”, but also because the author poses open problems, from Level 1 (*the author feels he could do, if he tried*) to Level 3 (*touching essential issues, with currently no way of telling how difficult they might be*).

The topic of this book is exemplified by its most-studied case, the Sherrington-Kirkpatrick (SK) model. Consider N “sites” and the 2^N “configurations” $\sigma = (\sigma_i,\ 1 \leq i \leq N) \in \{-1,1\}^N$. For any collection of $\binom{N}{2}$ real numbers $(g_{ij}, 1 \leq i < j \leq N)$ we can define a probability distribution on configurations via

$$P(\sigma|\{g_{ij}\}) \propto \exp\left(\frac{\beta}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + h \sum_{1\leq i\leq N} \sigma_i\right)$$

where $\beta, h \geq 0$ are further parameters. To interpret this formula, consider $\beta > 0, h = 0$. If $g_{ij} > 0$ then the sites i and j “prefer” to have the same state ± 1, whereas if $g_{ij} < 0$ then they prefer opposite states. If the values $\{g_{ij}\}$ are haphazardly negative or positive, it will be impossible to satisfy all these preferences, and a complicated probability distribution emerges from all these opposing and reinforcing interactions. Now suppose the values $\{g_{ij}\}$ were first picked as a realization of $\binom{N}{2}$ independent standard Normal random variables. While the probability distribution on configurations depends on the realization, one expects that in the $N \to \infty$ limit both qualitative properties, and quantitative statistical properties of the whole configuration (*observables* involving averages over sites, rather than the state of site 42 in particular) will be almost the same for all “typical” realizations $\{g_{ij}\}$.

Such models arose in statistical physics in the context of spin glasses, solids without regular structure. The SK model is a *mean-field* model, in that it ignores the geometry of 3-dimensional space. It was originally introduced as a toy model whose behavior should be easy to understand mathematically, but then extensive study by non-rigorous but ingenious methods of statistical physics revealed unexpectedly intricate structure.

This book seeks to start from basics (assuming little more than undergraduate probability and analysis, and in particular assuming no knowledge of statistical physics) and develop the rigorous theory, in almost complete detail, as far as possible. By focusing on a small set of models, the book manages to get close to the frontier of what is known rigorously about these models, though this often falls short of what has been discussed non-rigorously in the physics literature. Much of this rigorous work is due to Talagrand himself, though due credit is paid to others, Guerra in particular.

It is difficult to describe the mathematical content in a way that is useful for the intended novice audience. Analytically, a basic object of study is the *partition function*

$$Z_N = Z_N(\beta, h, \{g_{ij}\}) =$$
$$\sum_{\sigma\in\{-1,1\}^N} \exp\left(\frac{\beta}{\sqrt{N}} \sum_{1\leq i<j\leq N} g_{ij}\sigma_i\sigma_j + h \sum_{1\leq i\leq N} \sigma_i\right)$$

and one asks whether

$$N^{-1}\log Z_N \approx p_N(\beta,h) := E[N^{-1}\log Z_N] \qquad (1)$$

for typical $\{g_{ij}\}$, and for a formula for the limit

$$p_N(\beta,h) \to p(\beta,h).$$

Somewhat more probabilistically, it is natural to consider the *overlap*

$$R_{12} = \frac{1}{N}\sum_{i=1}^{N} \sigma_i^1\sigma_i^2$$

of two typical configurations (*replicas*) over the same $\{g_{ij}\}$. Calculations involving more than two replicas frequently arise. The *cavity method* is in one sense just induction on N, but is better described as a careful "bookkeeping" scheme for organizing complicated calculations relating N-site and $N-1$ site models. The *smart path method* derives inequalities between a simpler distribution and a more complicated distribution by continuous interpolation in a well-chosen way. *Concentration of measure* sometimes enables one to check (1) readily. These ideas arise near the start of the SK model analysis. A glimpse at the rest of the book is provided by some phrases and chapter titles: second moment calculations and the Almeida-Thouless line; Guerra's broken replica symmetry bound and the Parisi formula; the capacity of the perceptron (Ising case, Gaussian case, Spherical case); the Hopfield model; the p-spin interaction model at low temperature.

In summary, this is a book of ideas and calculations which build upon each other to create a rich and informative theory. Not a book for browsing, but one which demands concentrated study. Having these basic rigorous results in one place, with clear exposition of the techniques, will surely catalyze research into rigorous theory over the next few years.

It is intriguing to draw parallels between current work on spin glass models and the previous generation's work on percolation models, which also were studied in non-rigorous detail by physicists before attracting attention as rigorous mathematics [3]. The basic percolation models, just as the Sherrington-Kirkpatrick model, are highly over-simplified as models of real-world phenomena, but have turned out to be conceptually fundamental and useful for comparison purposes (e.g. "percolation substructures" [2]) in the study of more general real-world-inspired stochastic spatial models. It is reasonable to guess that these fundamental spin glass models and their analysis techniques will in future turn out to be equally pervasive.

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References

[1] P. Billingsley. *Convergence of Probability Measures.* Wiley, 1968.

[2] R. Durrett. *Lecture Notes on Particle Systems and Percolation.* Wadsworth, Pacific Grove CA, 1988.

[3] H. Kesten. *Percolation Theory for Mathematicians.* Birkhauser, 1982.