presented very clearly and intuitively, with some physical motivation, though with less rigor than some may prefer. The treatment of singularly perturbed two-point boundary value problems limits itself to linear second-order equations, but turning points are allowed, both at an endpoint and at an interior point. By using a WKB approach, Cheng is able to solve some examples with boundary layer resonance and two higher-order turning point problems that were presumably solved incorrectly by Bender and Orszag. The final chapter treats nonlinear oscillations. It generalizes the Poincaré–Linstedt procedure as a renormalization of angular frequency and two-timing as a renormalized two-scale method, and ends by illustrating the renormalization group method.

The subtleties addressed by Professor Cheng will be of interest to experts and more curious students who are up to handling the D (difficult) problems of Bender and Orszag.

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Random Fragmentation and Coagulation Processes. *By Jean Bertoin.* Cambridge University Press, Cambridge, UK, 2006. $65.00. viii+ 280 pp., hardcover. ISBN 978-0-521-86728-3.

As suggested by von Neumann's famous remark, "Mathematical ideas originate in empirics, although the genealogy is sometimes long and obscure," it is not always easy to identify a specific moment at which a definition-theorem-proof mathematical topic crystalizes out of more applied questions. But this gem of a monograph, written with a quintessentially French style and clarity, provides a self-contained account of a recently emerged topic that will be seen as a cornerstone for future theoretical developments.

The basic idea of *stochastic fragmentation* is to start with a unit mass particle. By time t this has split into particles of masses (x_i), where in the *conservative* case we require $\sum_i x_i = 1$ and more generally we require $\sum_i x_i \le 1$. Different particles evolve independently, a mass-x particle splitting at some stochastic rate λ_x into particles whose relative masses $(x_j/x,\ j \ge 1)$ follow some probability distribution $\mu_x(\cdot)$. (So the model neglects detailed three-dimensional geometry; the shape of a particle is assumed not to affect its propensity to split, and different particles do not interact.) Especially tractable is the *self-similar* case, where $\mu_x = \mu_1$ and $\lambda_x = x^\alpha$ for some *scaling exponent* α. Such processes are closely related to classical topics in theoretical and applied probability—the log-masses form a continuous-time branching random walk, and the mass of the particle containing a tagged atom forms a continuous-time Markov process on state space $(0, 1]$. Chapter 1 gives a clear description (assuming a first-year graduate knowledge of measure-theoretic probability at the level of [4], say) of the general setup, laws of large numbers for the mass spectrum of all particles and for the largest particle mass, the usefulness of additive martingales in the $\alpha = 0$ ("homogeneous") case, and the fact that for $\alpha < 0$ the mass disappears ("into dust") at some finite time even though the evolution equation appears mass-conserving.

One abstract, though conceptually simple and (where applicable) powerful, way

of thinking about $n \to \infty$ limits of random n-element structures is as follows. Within the n-element structure pick k random elements, look at the induced substructure on these k elements, and take a limit (in distribution) as $n \to \infty$ for fixed k. Within the limit random structures ($\mathcal{S}_k$, say) the k elements are exchangeable, and the distributions are consistent as k increases and therefore define an infinite structure $\mathcal{S}_\infty$ which for many purposes serves as an $n \to \infty$ limit of the original n-element structures. For instance, the historically second use of this method was in a formalization of *continuum random trees* as rescaled limits of finite random trees [1]. Historically the *first* use was by Kingman in his study of partitions on n genes (same site in different individuals) into allelic types. Here the limit space of interest is the space of unordered countable (x_i), where $x_i > 0$, $\sum_i x_i = 1$. An elementary formalization of this space uses the decreasing rearrangement of (x_i). Instead, the methodology above is to consider a partition of the unit interval into subintervals of lengths x_i, take independent uniform random variables $(U_1, \ldots, U_k)$ to construct a random partition $\mathcal{S}_k$ of $\{1, \ldots, k\}$ based on which U_i fall into common subintervals, then let $k \to \infty$ to obtain a random infinite exchangeable partition. The point is that a *random* unordered countable (x_i) has the same representation as an infinite exchangeable partition, and *Kingman's paintbox theorem* identifies the general infinite exchangeable random partition as arising in this way. Chapter 2 describes this general setup and the particular case of the two-parameter Poisson–Dirichlet distribution [6], which turns out to play a central role in several different aspects of random partition theory.

Chapter 3 returns to stochastic fragmentation. What allowed reduction (in Chapter 1) to classical results was the existence of a discrete genealogical tree, because particles survive for strictly positive time before splitting. But the most general notion of stochastic fragmentation allows continual splitting off of small masses; to handle this rigorously requires more sophistication, and here the "exchangeable random partition" setup in Chapter 2 turns out to be useful. In particular, Chapter 3 characterizes the general self-similar fragmentation process. This theory is closely analogous to the theory of Lévy processes, the subject (not coincidentally) of a previous book [3] by the author.

Coagulation (this word, introduced in German in [7], sounds strange to the native English speaker to whom it suggests blood clotting; *coalescence* seems a more apposite English word) concerns models of clusters joining together. Intuitively, coalescence and fragmentation are time-reversals of each other, and there are several fascinating examples of special cases where a precise duality relation exists and is useful, but somewhat surprisingly there seems to be no general precise duality relationship within the usual stochastic models.

One setting for stochastic coalescence is to have n particles, initially in single particle clusters, and let clusters merge according to a kernel $\kappa(x, x')$ indicating the rate (probability per unit time) at which a typical pair of clusters of sizes x and x' may merge. There is an intuitively natural *hydrodynamical limit*, that is, differential equations for the relative proportions $y_i(t)$ of clusters of size i in the $n \to \infty$ limit, and this *Smoluchowski coagulation equation* has a long history in several areas of science such as physical chemistry, as indicated in the survey [2]. Recent theoretical work has made rigorous the connection between the stochastic and deterministic models, and part of this is described in Chapter 5. Chapter 5 also describes the surprising connection between the additive ($\kappa(x, x') = x + x'$) stochastic case and a fragmentation process involving random trees, as well as the more obvious identification of the multiplicative case with the process of component sizes in a random graph process.

In population genetics, one can trace backwards in time the "genealogy" (lines of descent) of the alleles at a specific site of the genome in a population of size n. Under simplified assumptions (random mating and random numbers of offspring) the mean number of generations back to the most recent common ancestor is of order n. *Kingman's coalescent* describes the $n \to \infty$ limit process in rescaled time; each pair of lines of descent coalesces at constant stochastic rate 1. Chapter 4 describes this specific

process and expresses it more abstractly in terms of exchangeable random partitions. This leads to an elegant general theory of *exchangeable coalescents*, which eliminates the "only binary merging" aspect of Kingman's coalescent, and is interpretable as $n \to \infty$ limit genealogies of more general models in population genetics.

The theoretical probability research topics of this book have been developed over the last ten years, and while the text material is not intended to be at today's front line of research in this active topic, the comments at chapter ends give useful and remarkably complete references to subsequent research. Pitman's lecture notes [5] give a more compressed treatment of a broader range of topics (partly overlapping the topics of this book) relating probabilistic combinatorics to stochastic processes.

As befits the first theory monograph, there is a certain emphasis on abstract structure which gives this book a quite different style than previous expository writing on these topics, but it succeeds wonderfully as an authoritative account of the setup and the key results on fragmentation and coalescence.

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Differential Equations on Fractals: A Tutorial. *By Robert S. Strichartz.* Princeton University Press, Princeton, NJ, 2006. $29.95. xiv+169 pp., softcover. ISBN 978-0-691-12731-6.

Does a ragged rocky coastline hum when the sea churns against it?

We are all familiar with the clamped vibrating string and the associated eigenvalue problem. The string is supposed to be infinitely thin and its displacements $u(x)$ to be infinitesimally small, yet its mass and elasticity are finite. Elementary calculus applied to this model tells us that its behavior is governed by the Dirichlet eigenvalue problem $-\triangle u = \lambda u$ for $x \in [0,1]$ with $u(0) = u(1) = 0$, where $\triangle$ denotes the one-dimensional Laplacian d^2/dx^2.

But what you may not have thought about is that this classical problem contains self-similarity. For example, the real interval $[0,1]$ is the union of two similitudes, of scaling factor one-half, applied to it. Also, if G is the graph of an eigenfunction for eigenvalue λ, then $T_1(G) \cup T_2(G)$ is the graph of an eigenfunction for the eigenvalue 4λ, where $T_1(x,y) = (\frac{1}{2}x, \frac{1}{2}y)$ and $T_2(x,y) = (\frac{1}{2}x + \frac{1}{2}, -\frac{1}{2}y)$. Starting from such self-similarity information you can deduce properties of $\triangle$. For example, you can infer that the spectral density of the limit of the family of discrete Laplacians $\{\triangle_m\}_{m=1}^{\infty}$, where $\triangle_m$ is represented by the m-by-m matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & . & 0 \\ 0 & . & . & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

is the natural invariant measure on the Julia set for $z \to z(4-z)$. (Note that it is the sequence $\{4^m \triangle_m\}_{m=1}^{\infty}$ which converges to $\triangle$.) You can also gain new insights into the forms of the eigenfunctions $\{\sin n\pi x\}_{n=1}^{\infty}$, the Gauss–Green formulas, the spectral dimension, and so on.