Lecture 7: Game theory

David Aldous

September 14, 2016

STAT 155 is an entire course on Game Theory.

In this lecture we illustrate Game Theory by first focusing on one particular game for which we can get data. The game is relevant to one of the central ideas of game theory. Does the data – how people actually play the game – correspond roughly to what theory says?

We will do some math calculations "because we can" – more details in write-up. Continuing to analyze the data, or doing a simulation study of more complex strategies, would be a nice **course project**. Also, finding and studying some other observable online game-theoretic game would be a good **project**.

There are many introductory textbooks and less technical accounts of game theory – see write-up. Here is a 1-slide overview.

1. **Setting:** players each separately choose from a menu of actions, and get a payoff depending (in a known way) on all players' actions.
2. Rock-paper-scissors illustrates that one should use a **randomized** strategy, and so we assume a player's goal is to maximize their expected payoff. There is a complete theory of such two-person zero-sum games.
3. For other games, a fundamental concept is *Nash equilibrium* strategy: one such that, if all other players play that strategy, then you cannot do better by choosing some other strategy. This concept is motivated by the idea that, if players adjust their strategies in a selfish way, then strategies will typically converge to some Nash equilibrium.
4. More advanced theory is often devoted to settings where Nash equilibria are undesirable in some sense, as with Prisoners' Dilemma, and to understanding why human behavior is not always selfish.

A “survey”' **project** would be to look at the literature on game theory in some specific application field.

A (slightly simplified) math description of the actual game we shall study.

- There are 5 items of somewhat different known values, say $\{8, 7, 6, 5, 4\}$ dollars.
- There are 10 players.
- A player can make a sealed bid for (only) one item, during a window of time.
- During the time window, players see how many bids have already been placed on each item, but do not see the bid amounts.

When time expires each item is awarded to the highest bidder on that item. We assume players are seeking to maximize their expected gain.

So a player has to decide three things; when to bid, which item to bid on, and how much to bid.

The game is called *Dice City Roller* on `pogo.com`.

[show DCR in progress]

We get data by screenshots at 14, 5 and 0 seconds before deadline, and then after winning bids are shown.

Who are the actual players?

bbain_

Send Message A

Profile Badges Stats Gifts

Tokens:
49,933,882
Member Since: Apr 03, 2000

Club Time Rewards Program :

Club Time Rewards Year 9

View Mini Snapshots

Age	Gender
74	**female**

Location
al now/ moved from ca

Occupation
retired or just tired

Relationship Status
married

bbain_'s Guestbook

Write a comment...

Julia4719

Send Message A

Profile Badges Stats Gifts

Tokens:
6,399,469
Member Since: May 28, 2012

Club Time Rewards Program :

Club Time Rewards Year 1

"Are we having fun yet?"

Age	Gender
65	**female**

Location
TYLER,TX

Occupation
RETIRED

Relationship Status
married

BABY IT'S COLD OUTSIDE

View Mini Snapshots

We study math for simplified version without the time window – each player just places a sealed bid without knowledge of other players actions.

In this case we can calculate the Nash equilibrium strategy explicitly – for any number of players, and any number and values of items.

Start with the simplest setting: 2 players, 2 items of values 1 and b, where $0 < b \leq 1$.

We have a little data from playing this in Lecture 1.

[2014]

24/35 students bid on the $1, 11/35 bid on the 50c

bids on $1	bids on 50c
0	0 0
25	1
41	10
42	25
45	26
48	32
49 49 49 49 49 49	37
50 50 50 50	40
51	45
55	49
67	
75 75	
80	
81	
100	

Consider a person P who bid 49.
When we match with a random other person:

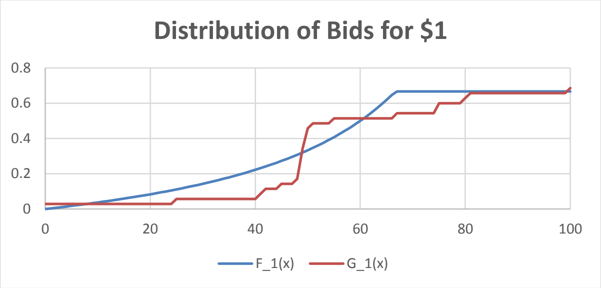
chance 12/34 P gains 0 (other person bid more on the $1)
chance 17/34 P gains 51 (= 100 − 49) (P won the bid)
chance 5/34 we do coin-toss to decide winner: P gains 51/2.

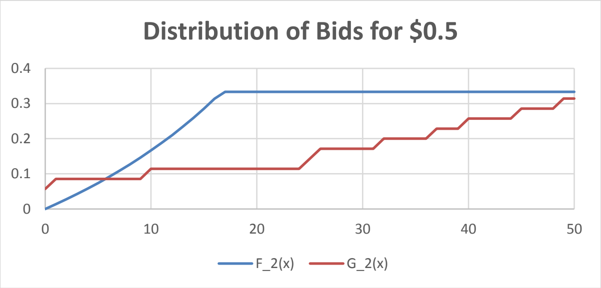
So P's expected gain = 29.25

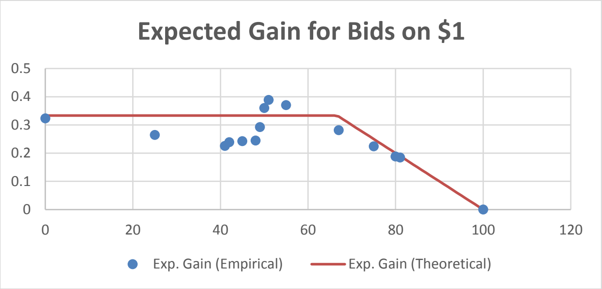
[2016]

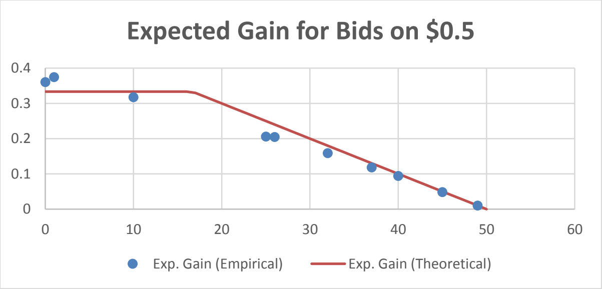
18/35 students bid on the $1, 17/35 bid on the 50c

bids on $1	bids on 50c
20	1 1
50 50 50 50 50	2
62	5
65	10
70 70 70	30 30
74	38
75 75	40 40
76	45 45
83	48 48 48
95	50 50
99	


Distribution of Bids for $1
0.8
0.6
0.4
0.2
0
0
20
40
60
80
100
F_1(x)
G_1(x)


Distribution of Bids for $0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
F_2(x)
G_2(x)


Expected Gain for Bids on $1
0.5
0.4
0.3
0.2
0.1
0
0
20
40
60
80
100
120
Exp. Gain (Empirical)
Exp. Gain (Theoretical)


Expected Gain for Bids on $0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
60
Exp. Gain (Empirical)
Exp. Gain (Theoretical)

For the in-class game, the winner was the student who bid 51c for the $1 item.

I will develop some math theory, first in this "two players, two items" setting. My point is to show that it's not terribly complicated. See write-up for more theory.

A player's strategy is a pair of functions (F_1, F_b):

$$F_1(x) = \mathbb{P}(\text{ bid an amount} \leq x \text{ on the first item}), \quad 0 \leq x \leq 1 \quad (1)$$
$$F_b(y) = \mathbb{P}(\text{ bid an amount} \leq y \text{ on the second item}), \quad 0 \leq y \leq b \quad (2)$$

where

$$F_1(1) + F_b(b) = 1. \quad (3)$$

We can equivalently work with the associated densities

$$f_1(x) = F_1'(x), \quad f_b(y) = F_b'(y).$$

Suppose your opponent's strategy is some function (f_1, f_b) and your strategy is some function (g_1, g_b).

What is the formula for your expected gain?
[do on board]

- opponent's strategy is (f_1, f_b), your strategy is (g_1, g_b).

Your expected gain is

$$\int_0^1 (1-x) g_1(x) [F_1(x) + F_b(b)]\, dx + \int_0^b (b-y) g_b(y) [F_b(y) + F_1(1)]\, dy. \quad (4)$$

We need an obvious fact [picture on board].
Given a payoff function $h(x) \geq 0$ with $h^* = \max_x h(x)$, consider the expected payoff $\int h(x) g(x) dx$ when we choose x according to a probability density g. Then we get the maximum expected payoff if and only if

$$h(x) = c \text{ for all } x \in \text{support}(g)$$

$$h(x) \leq c \text{ for all } x \notin \text{support}(g)$$

for some c (which is in fact h^*).

Now our expected gain (4) is of this form, thinking of (g_1, g_b) as a single probability density function. Apply the "obvious fact":

So given your opponent's strategy (f_1, f_b), your expected gain is maximized by choosing a strategy (g_1, g_b) satisfying, for some constant c

$$\begin{aligned}(1-x)[F_1(x)+F_b(b)] &= c \text{ on support}(g_1)\\ &\le c \text{ off support}(g_1)\\ (b-y)[F_b(y)+F_1(1)] &= c \text{ on support}(g_b)\\ &\le c \text{ off support}(g_b)\end{aligned}$$

Now the definition of (f_1, f_b) being a **Nash equilibrium** strategy is precisely the assertion that the (in)equalities above hold for $(g_1, g_b) = (f_1, f_b)$.

So now we have a set of equations for the NE strategy.

$$
\begin{aligned}
(1-x)[F_1(x)+F_b(b)] &= c \text{ on support}(f_1) & (5)\\
&\leq c \text{ off support}(f_1) & (6)\\
(b-y)[F_b(y)+F_1(1)] &= c \text{ on support}(f_b) & (7)\\
&\leq c \text{ off support}(f_b) & (8)
\end{aligned}
$$

with "boundary conditions"

$$F_1(0) = F_b(0) = 0; \; F_1(1) + F_b(b) = 1.$$

Note that in any game we can do some similar argument to get equations that a NE must satisfy. STAT 155, like most game theory, focusses on a discrete menu of actions – our example is continuous.

Theory talks about existence and uniqueness of solutions, for general games. We can just go ahead and solve these particular equations. The write-up shows how to solve "as math" without thinking about the game interpretation. The answer appears as

$$F_1(x) = \frac{b}{1+b}\left(\frac{1}{1-x} - 1\right) \text{ on } 0 \le x \le \frac{1}{1+b} \quad (9)$$

$$F_b(y) = \frac{1}{1+b}\left(\frac{b}{b-y} - 1\right) \text{ on } 0 \le y \le \frac{b^2}{1+b}. \quad (10)$$

The corresponding densities are

$$f_1(x) = \frac{b}{1+b}(1-x)^{-2} \text{ on } 0 \le x \le \frac{1}{1+b} \quad (11)$$

$$f_b(y) = \frac{b}{1+b}(b-y)^{-2} \text{ on } 0 \le y \le \frac{b^2}{1+b}. \quad (12)$$

The expected gain for each player works out as

$$\mathbb{E}[\text{gain}] = \frac{b}{1+b}.$$

[show figures again – how close is data to theory?]

An important general principle

> *If opponents play the NE strategy then any non-random choice of action you make in the support of the NE strategy will give you the same expected gain (which equals the expected gain if you play the random NE strategy), and any other choice will give you smaller expected gain.*

This "**constant expected gain**" principle is true because the NE expected gain is an average gain over the different choices in its support; if these gains were not constant then one would be larger than the NE gain, contradicting the definition of NE.

In our game, if you bid x on item 1, where x is in the support $0 \leq x \leq \frac{1}{1+b}$, then your chance of winning is (by calculation) $\frac{b}{1+b}(1-x)^{-1}$, so your expected gain is $(1-x) \times \frac{b}{1+b}(1-x)^{-1} = \frac{b}{1+b}$ as the general principle says.

Later we will use the general principle to calculate the NE for arbitrary numbers of players and prizes.

Note that the gap between your maximum bid and the item's value is the same for both items;

$$1 - 1/(1+b) = b - b^2/(1+b) = b/(1+b).$$

This follows from the "constant expected gain" principle above; if you bid the maximum value in the support the you are certain to win the item, so your gain must be the same for both items.

The same "equal gap principle" works by the same argument for general numbers of players and items (but is special to our particular game).

Consider the general case of $N \geq 2$ players and $M \geq 2$ items of values $b_1 \geq b_2 \geq \ldots \geq b_M > 0$. The bottom line (with a side condition – see write-up) is the formula

$$\mathbb{E}\text{ (gain to a player at NE) } = c = \left(\frac{M-1}{\sum_i b_i^{-1/(N-1)}} \right)^{N-1} \tag{13}$$

and the NE strategy is defined by the density functions

$$f_i(x) = \frac{M-1}{N-1} \frac{1}{\sum_j b_j^{-1/(N-1)}} (b_i - x)^{-N/(N-1)}, \quad 0 \leq x \leq b_i - c$$

for bids on prize i.

The next slide shows the main steps in the calculation.

Writing out the expression for the expected gain when you bid x_i on the i'th item, the "constant expected gain" property says

$$(b_i - x)\,(1 - (F_i(x_i^*) - F_i(x)))^{N-1} = c, \quad 0 \le x \le x_i^* := b_i - c \quad (14)$$

where $c =$ expected gain to a player at NE. Because a strategy is a probability distribution we have $\sum_i F_i(x_i^*) = 1$ and so

$$\sum_i (1 - F_i(x_i^*)) = M - 1.$$

Now using (14) with $x = 0$ we have

$$1 - F_i(x_i^*) = (c/b_i)^{1/(N-1)} \quad (15)$$

and so

$$\sum_i (c/b_i)^{1/(N-1)} = M - 1$$

identifying c.

The data from playing the game once in class doesn't fit the NE theory very well, but there's no reason it should fit. Recall our comment

- For other games, a fundamental concept is *Nash equilibrium* strategy: one such that, if all other players play that strategy, then you cannot do better by choosing some other strategy. This concept is motivated by the idea that, if players adjust their strategies in a selfish way, then strategies will typically converge to some Nash equilibrium.

The idea is that people "learn" after playing many times. In principle one could learn by recording your action and other players' actions in past games, and then use whatever strategy (probability distribution over actions) would have worked best (for you) in the past. Eventually you will reach an "equilibrium" in the sense of not changing strategy any longer.

In the actual *Dice City Roller* game people do play many times. How well does their actual play approximate the NE? This is the **anchor data** for this lecture.

There are two complications. The win amounts are actually random. Theory says only EVs should matter, and one expects players would "learn by experience" the EVs. In fact one can calculate them – thanks YuFan Hu.

Also the game imposes minimum bids. But the math analysis can be modified to handle this.

the true NE than to our approximate NE. A third possibility is described in section 6.1.

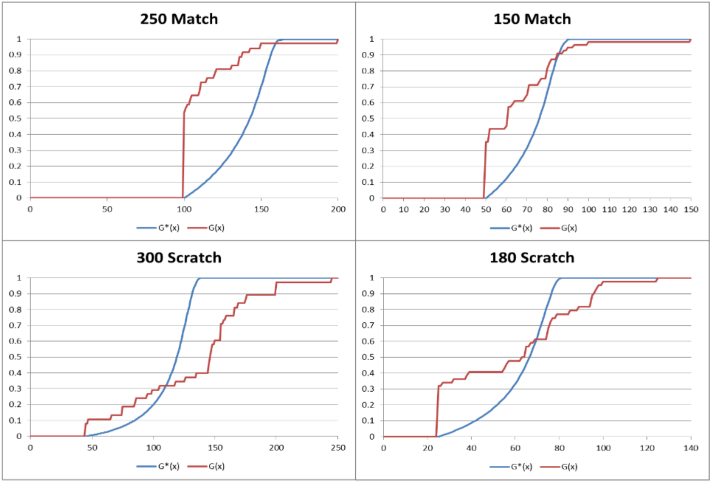


Figure 3. Comparison of winning bid distribution from data and from NE theory.

I don't want to spend all this lecture on the specifics of this game, but the "time window" aspect makes it more complicated and more (mathematically) challenging.

[say in words]

project: can you find another "observable" online game with a clear "game theory" component?

Conceptual digression. Prisoner's Dilemma reminds us that a NE is usually not a social optimum strategy. Here's another reminder, from Scott Aaronson's blog.

- *Why are even some affluent parts of the world running out of fresh water? Because if they weren't, they'd keep watering their lawns until they were.*
- *Why does it cost so much to buy something to wear to a wedding? Because if it didn't, the fashion industry would invent more extravagant "requirements" until it reached the limit of what people could afford.*

Again and again, I've undergone the humbling experience of first lamenting how badly something sucks, then only much later having the crucial insight that its not sucking wouldn't have been a Nash equilibrium.

Our other example is the **Least Unique Positive Integer** game. Each of N players chooses a number from $1, 2, 3, \ldots$. The winner is the person who chooses the smallest number that no-one else chooses.

- Quick to play with 5 - 40 people – need only pen and paper – organizer calls out to find winner.
- There might be no winner, but unlikely for large N.

Consider random strategy $\mathbf{p} = (p_1, p_2, \ldots)$. This is another game where we expect a unique NE and we could study data to see if real-world players adopt roughly the NE.

[show data from Lecture 1]

I outline [next slide] an easy approximate analysis of the NE, for reasonably large N. We expect the support of the NE to be $1 \leq i \leq K$ for some K depending on N.

If other players use $\mathbf{p}$ then

$$X_i = \text{number others choosing } i \approx \text{Poisson}(\lambda_i = (N-1)p_i)$$

The "constant expected gain" principle says that, whatever your choice of i in the support $[1, K]$, your chance of winning is $c \approx N^{-1}$. Choosing i, you win if no-one else chooses i and there is no unique chooser of any $j < i$, giving approximate equations

$$\mathbb{P}(X_i = 0, X_j \neq 1 \ \forall j < i) = 1/N, \ 1 \leq i \leq K.$$

For the left side there is a (complicated) formula in terms of $\mathbf{p}$. Can solve numerically. In particular, for $i = 1$ we see

$$\exp(-(N-1)p_1) \approx 1/N$$

and so

$$p_1 \approx \frac{\log N}{N-1}.$$

Then p_i decreases, slowly at first.

This game was played on a large scale, for large prizes, in Sweden, 7 times in 2007. Around 50,000 players. Analysis in paper *Testing game theory in the field: Swedish LUPI lottery games*.

[show ostling-figure, next slide]

Stopped after 7 weeks because it's possible to "cheat" with a coalition of players.

[explain how]

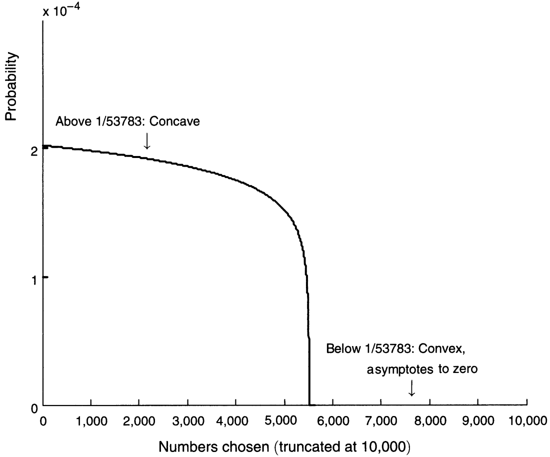


Figure 1. Poisson-Nash Equilibrium for the Field LUPI Game ($n = 53{,}783$, $K = 99{,}999$)