

Lecture 2: The Kelly criterion for favorable games: stock market investing for individuals

David Aldous

August 29, 2017

- Most adults drive/own a car
- Few adults work in the auto industry.

By analogy

- Most (middle class) adults will have savings/investments
- Few adults work in the banking/finance industries.

In my list of 100 contexts where we perceive chance

(22) Risk and reward in equity ownership
refers to the investor's viewpoint;

(81) Short-term fluctuations of equity prices, exchange rates etc
refers to the finance professional's viewpoint.

Over the last 30 years there has been a huge increase in the use of sophisticated math models/algorithms in finance, and many “mathematical sciences” majors seek to go into careers in finance. There are many introductory textbooks, such as Capinski - Zastawniak *Mathematics for Finance: An Introduction to Financial Engineering*. and college courses such as IEOR221. But these represent first steps toward a professional career; not really relevant to today's lecture.

Today's topic is “the stock market from the typical investor's viewpoint”. Best treatment is Malkiel's classic book *A Random Walk Down Wall Street*. But instead of summarizing that book, I will focus on one aspect, and do a little math.

There are many different theories/viewpoints about the stock market, none of which is the whole truth. So don't believe that anything you read is the whole truth.

- A conceptual academic view: financial markets are mostly about moving **risk** from those who don't want it to those who are willing to be paid to take over the risk.
- The rationalist view: today's stock price reflects consensus discounted future profits, plus a risk premium. This "explains" randomness mathematically (martingale theory). Textbook math theory starts by assuming some over-simplified random model without wondering how randomness arises.
- Many "psychological" theories say stock prices can stay out of alignment with "true value" for many years – cycles of sectors becoming fashionable/unfashionable.
- "Fundamental analysis" – see the Decal course *Introduction to Wall Street* – seeks to assess "true value" better than the market. The **efficient market hypothesis** says this is not practical.
- The "just a casino" view emphasizes the fact that most trading is from one owner of existing shares to another, rather than raising new capital for a business to start or grow.

Our starting point in this lecture

- The future behavior of the stock market will be statistically similar to the past behavior in some respects but will be different in other respects – and we can't tell which.

This is true but not helpful! So go in one of two directions.

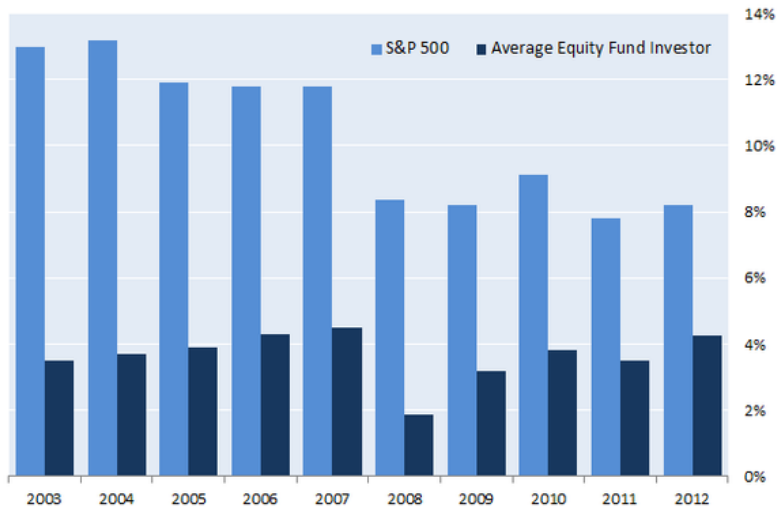
- 1 Devise your investment strategy under the assumption that “the future will be statistically similar to the past”, recognizing this isn't exactly true.
- 2 Decide (by yourself or advice from others) to believe that the future will be different from what the market consensus implies in certain specific ways, and base your strategy on that belief.

Wall St makes money mostly by (2), selling advice or speculating with their own money.

I'm not going to discuss whether you should invest in the stock market at all. If you choose to do so, here's the academic viewpoint.

- The default choice is (something like) a S&P index fund, available with very low expenses.
- As a matter of logic, because most investments are made via professional managers, their average gross return must be about the same as the market average, so the actual return to an individual investor must be on average be less than the market average, because managers charge fees and expenses.
- There's overwhelming empirical evidence (next slide, and **course project: survey the literature**) that individual investors on average do even worse, typically by going in and out of the market, or switching investments.

20-Year Returns: S&P 500 vs. Average Investor



Source: DALBAR's 2013 Quantitative Analysis of Investor Behavior

In class I ask students

Suppose you invest \$1,000 today in the stock market, more precisely in an S&P500 index fund. What do you guess the investment will be worth in 10 years?

Converting their answers to annual percentage growth, the spread of their guesses is shown here.

Table: Student guesses annual S&P500 growth next 10 years, and actual growth rate over previous 3 calendar years

date	25th percentile	median	75th percentile	previous
9/17/2008	-0.5%	5.4%	8.4%	6.6%
9/6/2011	-3.5%	4.1%	6.1%	- 5.0%
9/3/2014	4.2%	7.8%	11.2%	13.6%
1/20/2016	4.8%	8.2%	14.9%	12.7%
8/24/2017	3.4%	7.2%	19.6%	9.0%

This is roughly consistent with the often-remarked observation that amateur investors tend to base their expectations on the previous several years performance.

[show million dollar bet]

In comparing our default (index fund) with more sensible possible alternatives, the issue is measuring risk and reward in the stock market. Here is the “anchor” data for this lecture.

[show IFA page]

Implicit in the figure is that we measure

reward = long-term growth rate

risk = SD of percentage change each year.

Here is a first issue that arises.

*it is very hard to pin down a credible and useful number for the historical **long-term average growth rate** of stock market investments.*

Over the 50 years 1967-2017 the total return (including dividends) from the S&P500 index rose at (geometric) average rate 10.28% . This is from IFA – let's check another source. [show other source].

Aside from the (rather minor) point that we are using a particular index to represent the market what could possibly be wrong with using this figure? Well,

- it ignores expenses
- it is sensitive to choice of start and end dates; starting in 1950 would make the figure noticeably higher, whereas ending in 2009 would make it noticeably lower.
- to interpret the figure we need to compare it to some alternative investment, by convention some “risk-free” investment.
- it ignores inflation
- it ignores taxes.

Here are two graphs which give very different impressions of long-term stock market performance.

[show S&P index – click on max]

[show inflation-adjusted S&P]

In comparing our default (index fund) with more sensible possible alternatives, the issue is measuring **risk** and **reward** in the stock market. Regarding reward, as said above

*it is very hard to pin down a credible and useful number for the historical **long-term average growth rate** of stock market investments.*

But a typical conclusion is

Over the very long run, the U.S. stock market has had an inflation-adjusted annualized return rate of between six and seven percent.

It is not obvious how to measure – assign a numerical value to – the **risk** of an investment strategy. If I lend you \$100 today and you promise to pay me back \$200 in 10 years then my “reward” is 7.0% growth rate but my “risk” is that you don’t pay me back – and I don’t know that probability. A convention in the stock market context is to interpret risk as variability, and measure it as SD of annual percentage returns.

(**Project:** literature on alternative measures of risk?).

[Show link 2 again – annual returns]

IFA and similar sites start out by trying to assess the individual's subjective risk tolerance using a questionnaire. The site then suggests one of a range of 21 portfolios, represented on the slightly curved line in the figure. The horizontal axis shows standard deviation of annual return, (3% to 16%), and the vertical axis shows mean annual return (6% to 13%). Of course this must be historical data, in this case over the last 50 years. Notwithstanding the standard “past performance does not guarantee future results” legal disclaimer, the intended implication is that it is reasonable to expect similar performance in future.

- Questions:** (a) How does this relate to any theory?
(b) Should you believe this predicts the actual future if you invested?
(c) Does the reward/risk curve continue upwards further?

- Answers:** (a) The Kelly criterion says something like this curve must happen for different “good” investment strategies.
(b) Even if future is statistically similar to past, any algorithm will “overfit” and be less accurate at predicting the future than the past.
(c) No.

Expectation and gambling.

Recalling some basic mathematical setup, write $\mathbb{P}(\cdot)$ for probability and $\mathbb{E}[\cdot]$ for expectation. Regarding gambling, any bet has (to the gambler) some random profit X (a loss being a negative profit), and we say that an available bet is (to the gambler)

favorable if $\mathbb{E}[X] > 0$

unfavorable if $\mathbb{E}[X] < 0$

and *fair* if $\mathbb{E}[X] = 0$.

Note the word *fair* here has a specific meaning. In everyday language, the rules of team sports are *fair* in the sense of being the same for both teams, so the better team is more likely to win. For 1 unit bet on team B, that is a bet where you gain some amount b units if B wins but lose the 1 unit if B loses,

$$\mathbb{E}[\text{profit}] = bp - (1 - p); \quad p = \mathbb{P}(\text{B wins})$$

and so to make the bet is fair we must have $b = (1 - p)/p$. (Confusingly, mathematicians sometimes say “fair game” to mean each player has chance $1/2$ to win, but this is sloppy language).

Several issues hidden beneath this terminology should be noted. Outside of games we usually don't know probabilities, so we may not know whether a bet is favorable, aside from the common sense principle that most bets offered to us will be unfavorable to us.

One of these days in your travels, a guy is going to show you a brand-new deck of cards on which the seal is not yet broken. Then this guy is going to offer to bet you that he can make the jack of spades jump out of this brand-new deck of cards and squirt cider in your ear. But, son, do not accept this bet, because as sure as you stand there, you're going to wind up with an ear full of cider.

[spoken by Sky Masterson in *Guys and Dolls*, 1955].

The terminology (fair, favorable, unfavorable) comes from the law of large numbers fact that if one could repeat the same bet with the **same** stake independently, then in the long run one would make money on a favorable bet but lose money on an unfavorable bet. Such “long run” arguments ignore the issues of (rational or irrational) risk aversion and utility theory, which will be discussed later. In essence, we are imagining settings where your possible gains or losses are small, in your own perception.

Unfavorable bets.

Roughly speaking, there are two contexts in which we often encounter unfavorable bets. One concerns most activities we call *gambling*, e.g. at a casino, and the other concerns insurance. Regarding the former, mathematicians often say ridiculous things such as

Gambling against the house at a casino is foolish, because the odds are against you and in the long run you will lose money.

What's wrong is the *because*. Saying

Spending a day at Disneyland is foolish, because you will leave with less money than you started with

is ridiculous, because people go to Disneyland for entertainment, and know they have to pay for entertainment. And the first quote is equally ridiculous. Casino gamblers may have irrational ideas about chance and luck, but in the U.S. they typically regard it as entertainment with a chance of winning, not as a plan to make money. So it's worth being more careful and saying

*Gambling against the house at a casino and expecting to **make money** is foolish, because the odds are against you and in the long run you will lose money.*

The second context is that buying insurance is mathematically similar to placing an unfavorable bet – your expected gain is negative, because the insurance company wants to cover its costs and make a profit. But the whole point of buying insurance is risk aversion, so this needs to be treated in the setting of utility theory and psychology of probability (a later Lecture).

So where can I find a favorable bet?

The wisecrack answer “start your own casino or insurance company” is not so practical, but a variant of the latter is. For those who can, following the advice

increase your insurance deductibles to the maximum you can comfortably afford to lose

is a favorable bet, likely to save you money over a lifetime.

In this lecture we consider **investing in the stock market as mathematically similar to making a sequence of favorable bets** (and letting your winnings ride). Exactly *why* one could consider this a favorable bet could be debated endlessly – standard economic theory asserts that investors need to be rewarded for taking risk rather than using alternative risk-free investments, while empiricists observe that, in countries without anti-capitalist revolutions, the historical performance of stock markets actually has been better than those alternatives.

long term versus short term.

In everyday language, a job which will only last six months is a short term job; someone who has worked for a company for fifteen years is a long term employee. Joining a softball team for a summer is a short term commitment; raising children is a long term commitment. We judge these matters relative to human lifetime; *long term* means some noticeable fraction of a lifetime.

Table: Effect of 7% interest, compounded annually.

year	0	4	8	12	16	20
simple interest	1000	1,280	1,560	1,840	2,120	2,400
compound interest	1000	1,311	1,718	2,252	2,952	3,870

One of several possible notions of *long term* in financial matters is “the time span over which compounding has a noticeable effect”. Rather arbitrarily interpreting “noticeable effect” as “10% more” and taking the 7% interest rate, this suggests taking 8 years as the cut-off for *long term*. Being about 10% of a human lifetime, this fortuitously matches reasonably well the “noticeable fraction of a lifetime” criterion above. And indeed in matters pertaining to individuals, financial or otherwise, most writers use a cut-off between 5 and 10 years for “long term”

- The mathematical theme of this lecture is the nature of compounding when gains and losses are unpredictable.
- The relevant arithmetic is **multiplication** not addition: a 20% gain followed by a 20% loss combine to a 4% loss, because $1.2 \times 0.8 = 0.96$.

Let's move on to some mathematics

First let us make explicit the type of model used implicitly above. A “return” $x = 0.2$ or $x = -0.2$ in a year means a 20% gain or a 20% loss.

The IID model.

Write X_i for the return in year i . Suppose the (X_i) are IID random variables. Then the value Y_n of your investment at the end of year n is

$$Y_n = Y_0 \prod_{i=1}^n (1 + X_i) \quad (1)$$

where Y_0 is your initial investment.

Conceptually, we are assuming the future is statistically the same as the past, and assuming independence over different time periods.

To analyze this model we take logs and divide by n :

$$n^{-1} \log Y_n = n^{-1} \log Y_0 + n^{-1} \sum_{i=1}^n \log(1 + X_i)$$

and the law of large numbers says that as $n \rightarrow \infty$ the right side converges to $\mathbb{E}[\log(1 + X)]$. We want to compare this to an investment with a non-random return of r . For such an investment (interest rate r , compounded annually) we would have $Y_n = Y_0(1 + r)^n$ and therefore $n^{-1} \log Y_n \rightarrow \log(1 + r)$. Matching the two cases gives the conclusion

In the IID model, the long term growth rate is

$$\exp(\mathbb{E}[\log(1 + X)]) - 1.$$

The formula looks strange, because to compare with the IID annual model we are working with the equivalent “compounded annually” interest rate. It is mathematically nicer to use instead the “compounded instantaneously” interest rate, which becomes just $\mathbb{E}[\log(1 + X)]$.

The main impact of this result is that what matters “in the long term” about the random return X is **not** precisely the mean $\mathbb{E}[X]$, but rather its “multiplicative” analog $\mathbb{E}[\log(1 + X)]$. Let us note, but set aside for a while, the points

- Is the model realistic for stock market investing?
- The phrase **long term** here refers to the applicability of the law of averages as an approximation to finite time behavior – this is a third meaning of the phrase, logically quite distinct from the two previous meanings.

Instead we focus on the conceptual point is that there are many investment possibilities, that is ways to allocate money to different risky or safe assets. Write α for a **portfolio**, that is a way of investing given proportions of your fortune in different assets.

We can now jump to our first key mathematical point.

The Kelly Criterion. *Suppose you have a range of possible investment portfolios α , which will produce return X_α , with known distribution. Then (assuming the IID model) the portfolio α that maximizes the long term growth rate is the portfolio that maximizes $\mathbb{E}[\log(1 + X_\alpha)]$. So choose that portfolio.*

The math here is just STAT134, but the implications are rather subtle, as we will see from playing with some hypothetical, very simple models.

Mathematics of the Kelly criterion: one risky and one safe asset.

Suppose there is both a “risky” (random return) asset (a “stock”, more realistically a S&P500 index fund) and a risk-free alternative investment (a “bond”) that pays a fixed interest rate r .

Suppose we choose some number $0 \leq p \leq 1$ and at the start of each year we invest a proportion p of our total “investment portfolio” in the stock market, and the remaining proportion $1 - p$ in the bond. In this case our return in a year is

$$X^* = pX + (1 - p)r$$

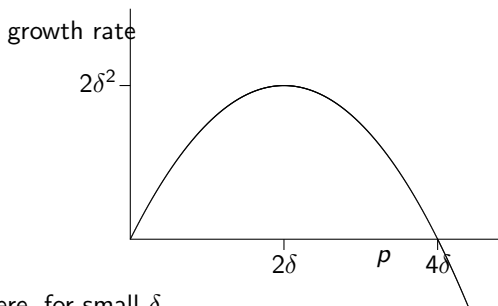
where X is the return on the stock. The long term growth rate is now a function of p : in the continuous setting

$$\text{growth}(p) = \mathbb{E}[\log(1 + pX + (1 - p)r)]. \quad (2)$$

The Kelly criterion says: choose p to maximize $\text{growth}(p)$. Let's see two examples. In the first X is large, and we end up with p small; in the second X will be small, and we end up with large p . In these two examples we take the time unit to be 1 day instead of 1 year (which doesn't affect math formulas).

Example: pure gambling.

Imagine a hypothetical bet which is slightly favorable. Suppose each day we can place a bet of any size s ; we will either gain s (with probability $0.5 + \delta$) or lose s (with probability $0.5 - \delta$), independently for different days (here δ is assumed small). Take $r = 0$ for the moment. What proportion p of our portfolio do we want to bet each day?



Here, for small δ ,

$$\begin{aligned}
 \mathbb{E}[\log(1 + pX)] &= \left(\frac{1}{2} + \delta\right) \log(1 + p) + \left(\frac{1}{2} - \delta\right) \log(1 - p) \\
 &\approx \left(\frac{1}{2} + \delta\right)(p - p^2/2) + \left(\frac{1}{2} - \delta\right)(-p - p^2/2) \\
 &= 2\delta p - p^2/2.
 \end{aligned}$$

Thus the asymptotic growth rate is approximately the quadratic function of p

$$G(p) = 2\delta p - p^2/2 \quad (3)$$

shown in the Figure. The Kelly criterion says to choose $p \approx 2\delta$ and then your long term growth rate will be $\approx 2\delta^2$.

Now recall that we simplified by taking $r = 0$; when $r > 0$, the fact that a proportion $1 - p \approx 1$ of the portfolio not put at risk each day can earn interest, brings up the optimal growth rate to $r + 2\delta^2$; the quantity $2\delta^2$ represents the *extra* growth one can get by exploiting the favorable gambling opportunity.

To give a more concrete mental picture, suppose $\delta = 1\%$. The model matches either of the two following hypothetical scenarios.

- (a) To attract customers, a casino offers (once a day) an opportunity to make a roulette-type bet with a 51% chance of winning.
- (b) You have done a statistical analysis of day-to-day correlations in some corner of the stock market and have convinced yourself that a certain strategy (buying a portfolio at the start of a day, and selling it at the end) replicates the kind of favorable bet in (a).

In either scenario, the quantity $2\delta^2 = 2/10,000$ is the “extra” long term growth rate available by taking advantage of the risky opportunity. Note this growth rate is much smaller than 2% “expected gain” on one bet. On the other hand we are working “per day”, and in the stock market case there are about 250 days in a year, so the growth rate becomes about 5% per year; recalling this is “5% above the risk-free interest rate”, it seems a rewarding outcome. But if δ were instead 0.5% then the extra growth rate becomes $1\frac{1}{4}\%$, and (taking into account transaction costs and our work) the strategy hardly seems worth the effort.

Implicit in the Figure (back 2 pages) is a fact that at first strikes everyone as counter-intuitive. The curve goes negative when p increases above approximately 4δ . So even though it is a favorable game, if you are too greedy then you will lose in the long run!

This setting was artificially simple; here is a first step towards a more realistic setting.

Example: what proportion of your portfolio to put into the “stock”?

As before, suppose there is both a “risky” (random return) asset (a “stock”, more realistically a S&P500 index fund) and a risk-free alternative investment (a “bond”) that pays a fixed interest rate r . Now we imagine typical values of the return X as ± 0.2 ., on a **yearly** time-scale. As before we choose some number $0 \leq p \leq 1$ and at the start of each year we invest a proportion p of our total “investment portfolio” in the stock market, and the remaining proportion $1 - p$ in the bond. We know that the long term growth rate is a function of p :

$$\text{growth}(p) = \mathbb{E}[\log(1 + pX + (1 - p)r)].$$

There is a nicer algebraic way of dealing with the interest rate r . Set

$$X = r + (1 + r)X^*$$

and interpret $X^* = (X - r)/(1 + r)$ as “return relative to interest rate”. Then a couple of lines of algebra let us rewrite the formula as

$$\text{growth}(p) = (1 + r) \exp(\mathbb{E}[\log(1 + pX^*)]) - 1 \quad (4)$$

and the optimization problem now doesn't involve any r .

If we imagine the stock market on a **daily** time-scale and suppose changes X^* are small, with mean μ and variance σ^2 , then we can use the series approximation

$$\log(1 + pX^*) \approx pX^* - \frac{1}{2}(pX^*)^2$$

to calculate

$$\mathbb{E}[\log(1 + pX^*)] \approx p\mu - \frac{1}{2}p^2(\mu^2 + \sigma^2) \approx p\mu - \frac{1}{2}p^2\sigma^2$$

(the latter because μ and σ^2 are in practice of the same order, so μ^2 is of smaller order than σ^2). So the Kelly criterion says: choose p to maximize $p\mu - \frac{1}{2}p^2\sigma^2$, that is choose

$$p = \mu/\sigma^2. \tag{5}$$

This is another remarkable formula, and let us discuss some of its mathematical implications.

1. The formula is (as it should be) *time-scale free*. That is, writing $\mu_{day}, \mu_{year}, \sigma_{day}^2, \sigma_{year}^2$ for the means and variances over a day and a N -day year, then (because compounding has negligible effect over a year) $\mu_{year} \approx N\mu_{day}$ and $\sigma_{year}^2 \approx N\sigma_{day}^2$, so we get the same value for μ/σ^2 whether we work in days or years.
2. Even though we introduced the setup by stating that $0 \leq p \leq 1$, the model and its analysis make sense outside that range. Economic theory and experience both say that the case $\mu < 0$ doesn't happen (investors are risk averse and so would buy no stock; this would cause the current price of stock to drop), but if it did then the formula $p = \mu/\sigma^2 < 0$ says that not only should we invest 100% of our wealth in the bond, but also we should "sell short" (i.e. borrow) stock and invest the proceeds in the bond.
3. More interesting is the case $p > 1$. Typical values given for the S&P500 index (as noted earlier, stating meaningful historical values is much harder than one might think) are (interest-rate-adjusted) $\mu = 5.6\%$ and $\sigma = 20\%$, in which case the Kelly criterion says to invest a proportion $p = 140\%$ of your wealth in the stock market, i.e. to borrow money (at fixed interest rate r) and invest your own and the borrowed money in the stock market.

What about the not-so-long term?

We started with the multiplicative model, which assumes that returns in different time periods are IID. This is not too realistic, but the general idea behind the Kelly criterion works without any such assumption, as we now explain.

Going back to basics, the idea

to invest successfully in the stock market, you need to know whether the market is going to go up or go down

is just wrong. Theory says you just need to know the *probability distribution* of a future return. So suppose (a very big **SUPPOSE**, in practice!) at the beginning of each year you could correctly assess the probability distribution of the stock return over the coming year, then you can use the Kelly criterion (2) to make your asset allocation. The fact that the distribution, and hence your asset allocation, would be different in different years doesn't make any difference – this strategy is still optimal for long-term growth.

The second key insight from mathematics

The numbers for growth rates that come out of the formula of course depend on the distributions of each next year's returns, but there's one aspect which is "universal". In any situation where there are sensible risky investments, following the Kelly strategy means that you accept a short-term risk which is always of the same format:

40% chance that at some time your wealth will drop to only 40% of what you started with.

The magical feature of this formula is that the percents always match: so there is a 10% chance that at some time your wealth will drop to only 10% of what you started with.

The math here is from the theory of *diffusions* or stochastic calculus; a little too hard to explain here.

x% chance that at some time your wealth will drop to only x% of what you started with.

For an individual investor, it is perfectly OK to be uncomfortable with this level of medium-term risk and to be less *aggressive* (in investment jargon) by using a *partial Kelly* strategy, that is using some smaller value of

p = proportion of your assets invested in stocks

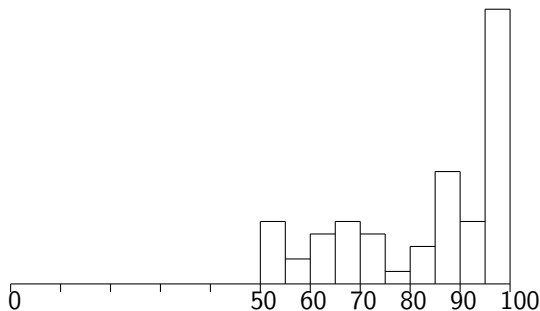
than given by the Kelly criterion. Theory predicts you will thereby get slower long-term growth but with less short-term volatility.

A memorable quote

The Kelly strategy marks the boundary between aggressive and insane investing.

How one might expect this theory (based on assuming known true probability distributions for the future, and on seeking to optimize long-term growth rate) to relate to the actual stock market is not obvious, but one can certainly look at what the actual percentages have been.

The next figure shows the historical distribution (based on hypothetical purchase of S&P500 index on first day of each year 1950-2009 and on subsequent monthly closing data) of the minimum future value of a 100 investment.



This is obviously very different from the “Kelly” prediction of a flat histogram over $[0, 100]$. This data is not adjusted for inflation or for comparison with a risk-free investment, and such adjustments (**course project?**) would make the histogram flatter, but still not close to the Kelly prediction. We mentioned before that over the historical long term it has been more profitable to borrow to invest more than 100% of your assets in the market. Both observations reflect the fact that the stock market fluctuates *less* than would the fortune of a Kelly-optimizing speculator.

Wrap up

The math I've shown is still a long way away from explaining the IFA graphic. They have many different "asset classes", each represented by an index fund [show IFA]. We want to construct a portfolio by weighting over each asset class (note we still avoid individual stocks). Any possible portfolio has (historical data) some average and some SD of annual return. We just choose a range of portfolios that maximize average for given SD (this exploits past pattern of correlations as well as averages and SDs).

Note that WolframAlpha will do this kind of thing for you [show].

The standard textbook math is *Modern portfolio theory* which is essentially the short-term Normal approximation for price fluctuations. The reason I emphasize Kelly instead is that (in principle) you could model "Black Swans" (rare severe shocks) or at least use actual historic annual return distributions rather than assuming log-Normal.

I'm not going to discuss whether you should invest in the stock market at all. If you choose to do so, here's the academic viewpoint.

- The default choice is (something like) a S&P index fund, available with very low expenses.

If you really want to do better then there are three vaguely plausible ways **none of which I recommend**.

- Borrow money, if you can find a low interest rate.
- Diversified Kelley-optimized portfolio, like IFA.
- Long term market timing such as the Shiller PE ratio [show]

Further reading

The two books most related to our “Kelly criterion” are

William Poundstone *Fortune's Formula*: a history of precisely this topic

Aaron Brown *Red-Blooded Risk: The Secret History of Wall Street*: his experience of 1980s risk management in finance.

A memorable quote from Brown: Kelly enables you *to get rich exponentially slowly*.

Any book by Robert Schiller.

For a little more math see the paper *Good and bad properties of the Kelly criterion*.