The Rejection Method

Suppose we have a distribution with known p.d.f. $f(x)$, from which we would like to generate random numbers. Further suppose that there is no simple expression for the inverse of the c.d.f. for this function, so that the probability integral transform is not an option. Then the following method, initially proposed by von Nuemann in 1951, and known as the rejection technique, proceeds as follows:

1. Find some function $g(x)$ which is greater than $f(x)$ at all points of interest. The function $g(x)$ is known as a *majorizing function*. Convert this function to a p.d.f., $f^*(x)$, through the following relation:

   $$f^*(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x)}$$

   (The limits of the integral may change depending on the support of the majorizing function.)

   The majorizing function should be integrable and invertible, so that a random sample from $f^*(x)$ is easily obtainable through the probability integral transform.

2. Generate a random variable X from $f^*(x)$.

3. Generate a random variable Y from a Uniform(0,1) distribution. Note that

   $$f_Y(y) = 1 \qquad 0 < y < 1,$$

   and

   $$F_Y(y) = y \qquad 0 < y < 1,$$

   where $f_Y(y)$ and $F_Y(y)$ are the p.d.f. and c.d.f., respectively of Y.

4. Let

   $$h(X) = \frac{f(X)}{g(X)}.$$

   If $Y <= h(X)$, set $Z = X$. Otherwise, go back to step 2.

Then the variable Z had p.d.f. $f(x)$.

Proof:

$$\begin{aligned}
P(Z <= r) &= P(X <= r | Y <= h(x)) \\
&= \frac{\int_{-\infty}^{r} \int_0^{h(x)} f^*(x) dy\, dx}{\int_{-\infty}^{\infty} \int_0^{h(x)} f^*(x) dy\, dx} \\
&= \frac{\int_{-\infty}^{r} f^*(x) \frac{f(x)}{g(x)} dx}{\int_{-\infty}^{\infty} f^*(x) \frac{f(x)}{g(x)} dx}
\end{aligned}$$

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\begin{aligned}
&= \frac{\int_{-\infty}^{r} \frac{g(x)}{\int_{-\infty}^{\infty} g(u)du} \cdot \frac{f(x)}{g(x)} dx}{\int_{-\infty}^{\infty} \frac{g(x)}{\int_{-\infty}^{\infty} g(u)du} \cdot \frac{f(x)}{g(x)} dx} \\
&= \int_{-\infty}^{r} f(x)dx,
\end{aligned}
$$

which is the c.d.f. of a random variable with p.d.f. $f(x)$, the desired result.