and taking the second mixed partial derivative makes it clear that the density function factors. On the other hand, if the density function factors, then the joint cdf can be expressed as a product:

$$\begin{aligned} F(x, y) &= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(u) f_Y(v)\, dv\, du \\ &= \left[\int_{-\infty}^{x} f_X(u)\, du\right] \left[\int_{-\infty}^{y} f_Y(v)\, dv\right] = F_X(x) F_Y(y) \end{aligned}$$

It can be shown that the definition implies that if X and Y are independent, then

$$P(X \in A, Y \in B) = P(X \in A) P(Y \in B)$$

It can also be shown that if g and h are functions, then $Z = g(X)$ and $W = h(Y)$ are independent as well. A sketch of an argument goes like this (the details are beyond the level of this course): We wish to find $P(Z \leq z, W \leq w)$. Let $A(z)$ be the set of x such that $g(x) \leq z$, and let $B(w)$ be the set of y such that $h(y) \leq w$. Then

$$\begin{aligned} P(Z \leq z, W \leq w) &= P(X \in A(z), Y \in B(w)) \\ &= P(X \in A(z)) P(Y \in B(w)) \\ &= P(Z \leq z) P(W \leq w) \end{aligned}$$

EXAMPLE **A** Suppose that the point (X, Y) is uniformly distributed on the square $S = \{(x, y) \mid -1/2 \leq x \leq 1/2, -1/2 \leq y \leq 1/2\}$: $f_{XY}(x, y) = 1$ for (x, y) in S and 0 elsewhere. Make a sketch of this square. You can visualize that the marginal distributions of X and Y are uniform on $[-1/2, 1/2]$. For example, the marginal density at a point x, $-1/2 \leq x \leq 1/2$ is found by integrating (*summing*) the joint density over the vertical line that meets the horizontal axis at x. Thus, $f_X(x) = 1, -1/2 \leq x \leq 1/2$ and $f_Y(y) = 1$, and $-1/2 \leq y \leq 1/2$. The joint density is equal to the product of the marginal densities, so X and Y are independent. You should be able to see from our sketch that knowing the value of X gives no information about the possible values of Y. ■

EXAMPLE **B** Now consider rotating the square of the previous example by 90° to form a diamond. Sketch this diamond. From the sketch, you can see that the marginal density of X is nonnegative for $-1/2 \leq x \leq 1/2$ as before, but it is not uniform, and similarly for the marginal density of Y. Thus, for example, $f_X(.9) > 0$ and $f_Y(.9) > 0$. But from the sketch you can also see that $f_{XY}(.9, .9) = 0$. Thus, X and Y are not independent. Finally, the sketch shows you that knowing the value of X— for example, $X = .9$— constrains the possible values of Y. ■