Martingale coupling of cumulative hazard and exponential variables by Azéma-Yor embedding in Brownian motion

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Cumulative Hazard Variables

- $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq$
- $N \geq 1$ a stopping time: $(N \leq n) \in \mathcal{F}_n$
- $h_n := P(N = n \,|\, \mathcal{F}_{n-1}), n \geq 1$
- $A_n := \sum_{k=1}^{n} h_k \uparrow A_\infty = A_N =$ *total hazard*

Key Facts:

- $A_n - 1(N \leq n)$ is an $(\mathcal{F}_n)$-martingale
- $\rightarrow A_\infty - 1$ as $n \rightarrow \infty$ assuming $P(N < \infty) = 1$
- Limit holds in L^p for $p \geq 1$
- $EA_\infty = 1$
- $EA_\infty^p < \Gamma(p+1)$ for $p > 1$.
- $EA_\infty^p > \Gamma(p+1)$ for $0 < p < 1$.

(cf. Dellacherie-Meyer Probabilités et Potentiel B (1982) §106)

Example: birthday repeat time

Birthday problem with y days/year.

- $Y_1, Y_2, \ldots$ independent uniform on $\{1, \ldots, y\}$
- $N := \min\{n : Y_n \in \{Y_1, \ldots, Y_{n-1}\}\}$
- $h_n := P(N = n \,|\, Y_1, \ldots, Y_{n-1}) = \frac{n-1}{y} 1(N \geq n)$

$A_n := h_1 + \cdots + h_n \qquad \Rightarrow$

- $A_\infty = \frac{1}{y}(0 + 1 + \cdots + (N-1)) = \frac{N(N-1)}{2y} \approx \frac{N^2}{2y}$
- $E[N(N-1)] = 2y \qquad \Rightarrow N$ is of order $\sqrt{y}$
- Simple formula for $E[N(N-1)]$ not so obvious from
- $P(N \geq n) = \left(1 - \frac{1}{y}\right) \cdots \left(1 - \frac{n-1}{y}\right) \approx \exp\left(-\frac{n^2}{2y}\right)$
- $P(N/\sqrt{y} \geq x) \to e^{-x^2/2}$ as $y \to \infty, \qquad x \geq 0$

Let ε be standard exponential: $P(\varepsilon > t) = e^{-t}, t \geq 0$. As $y \to \infty$:

- $\sqrt{2A_\infty} \approx N/\sqrt{y} \xrightarrow{d} \sqrt{2\varepsilon} \qquad$ and $\qquad A_\infty \xrightarrow{d} \varepsilon$

Compensators

Extend to continuous time: $(\mathcal{F}_t)_{t\geq 0}$, $(T \leq t) \in \mathcal{F}_t$.
We know (Doob-Meyer): $\exists!$ predictable $\uparrow$ $(A_t, t \geq 0)$ so that

- $A_t - 1(T \leq t)$ is an $(\mathcal{F}_t)$-martingale.
- $\to A_\infty - 1 = A_T - 1$ in every L^p if $P(T < \infty) = 1$.

Question:

- What can be said about the laws of such total hazard variables A_∞?

Neveu's Inequality

From Neveu *Martingales a temps discret* (1972):
(A_t) an $(\mathcal{F}_t)$-predictable $\uparrow$ process with $A_0 = 0$, $E(A_\infty) = 1$.

$$Z_t := E[A_\infty - A_t \,|\, \mathcal{F}_t] = E[A_\infty \,|\, \mathcal{F}_t] - A_t \qquad (\geq 0 \text{ super MG })$$

Suppose $0 \leq Z \leq 1$ (bounded potential). e.g. the *Azéma supermartigale* $Z_t := P(T > t \,|\, \mathcal{F}_t)$ for some random T [$(\mathcal{F}_t)$-stopping? $\mathcal{F}_\infty$-meas.?].

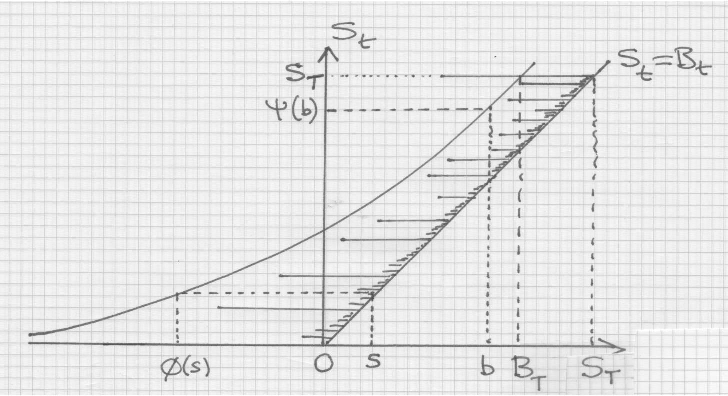
$$M_t := E[A_\infty \,|\, \mathcal{F}_t] = A_t + Z_t \geq 0 \qquad [\text{ UI MG }]$$

Note $M_0 = 1, \qquad M_\infty = A_\infty$.
Let $\tau_a := \inf\{t : A_t > a\}$. Then $(A_\infty > a) = (\tau_a < \infty) \in \mathcal{F}_{\tau_a -}$. So

$$\begin{aligned} E[A_\infty \,|\, A_\infty > a] &= E[A_{\tau_a -} \,|\, A_\infty > a] + E(Z_{\tau_a -} \,|\, A_\infty > a] && (1) \\ &\leq a + 1 && (2) \\ &= E(\varepsilon \,|\, \varepsilon > a) \text{ where } P(\varepsilon > t) = e^{-t}, t > 0. && (3) \end{aligned}$$

Conclusion: $A_\infty \leq_{\mathrm{mrl}} \varepsilon$ and $E(A_\infty) = E(\varepsilon) = 1$
Distribution of A_∞ is NBUE - Barlow-Proshan(1965)
Daley 1988 - Tight bounds in exponential approximation. Mark Brown, ...

The Azéma-Yor Construction



- $\psi(b) \uparrow, \quad \psi(-\infty) = 0, \quad \phi := \psi^{-1}.$
- $T := \inf\{t : S_t \geq \psi(B_t)\}, \qquad S_t := \sup_{0 \leq s \leq t} B_s$
- $P(B_T \geq b) = P(S_T \geq \psi(b)) = \exp\left(-\int_0^{\psi(b)} \frac{dy}{y - \phi(y)}\right)$

Azéma-Yor Calculus

- $\psi(b) \uparrow$, $\psi(-\infty) = 0, \phi := \psi^{-1}$.
- $T := \inf\{t : S_t \geq \psi(B_t)\}, \quad S_t := \sup_{0 \leq s \leq t} B_s$
- $\bar{F}(b) := P(B_T \geq b) = P(S_T \geq \psi(b)) = \exp\left(-\int_0^{\psi(b)} \frac{dy}{y - \phi(y)}\right)$
- Assume $f(b) := -\bar{F}'(b)$ exists, and use $\phi(\psi(b)) = b$

$$f(b) = \bar{F}(b) \frac{\psi'(b)}{\psi(b) - b}$$

$$\frac{d}{db}\left[\bar{F}(b)\psi(b)\right] = -bf(b)$$

$$\bar{F}(b)\psi(b) = \int_b^\infty x f(x)\,dx$$

$$\psi(b) = \frac{\int_b^\infty x f(x)\,dx}{\bar{F}(b)} = E[X \mid X \geq b] \text{ for } X = B_T$$

Azéma-Yor Theorem

For a distribution of X with $E(|X|) < \infty$, define the *barycenter function*

$$\psi_X(b) := E(X \mid X \geq b) \qquad [\quad = b \text{ if } P(X \geq b) = 0].$$

Theorem (Azéma-Yor (1979))

Let B be Brownian motion. For X with $E(X) = 0$ let $T := \inf\{t : S_t \geq \psi_X(B_t)\}$ where $S_t := \sup_{0 \leq s \leq t} B_s$. Then $B_T \stackrel{d}{=} X$ and $(B_{t \wedge T}, t \geq 0)$ is a uniformly integrable martingale.

Corollary: [Dubins-Gilat(1978)]

If (M_t) is a right-continous UI MG with $M_\infty \stackrel{d}{=} X$ then $\sup_t M_t \leq_{\rm st} S_T$.
Many variations and extensions now known:
See Obloj (2004) Probability Surveys + 129 citations.

Stochastic orders

Reference: Shaked and Shanthikumar(2007), *Stochastic orders*.

The *stochastic order*: $X \leq_{\rm st} Y \Leftrightarrow$

(i) $E\phi(X) \leq E\phi(Y) \quad \forall \phi \geq 0, \uparrow$;

(ii) $P(X > a) \leq P(Y > a)$ for all real a;

(iii) $\exists\ X'$ and Y' with $X' \stackrel{d}{=} X$, $Y' \stackrel{d}{=} Y$ and $P(X' \leq Y') = 1$.

The *convex order*: For integrable X and Y: $X \leq_{\rm cx} Y \Leftrightarrow$

(i) $E\phi(X) \leq E\phi(Y) \quad \forall$ convex ϕ

(ii) $E(X) = E(Y)$ and $E(X - a)_+ \leq E(Y - a)_+$ for all real a

(iii) $E(X) = E(Y)$ and $E|X - a| \leq E|Y - a|$ for all real a

(iv) $\exists X'$ and Y' with $X' \stackrel{d}{=} X$, $Y' \stackrel{d}{=} Y$ and $E(Y' \mid X') = X'$.

Mean residual life order

The *mean residual life order*: For integrable X and Y: $X \leq_{\mathrm{mrl}} Y \Leftrightarrow$

(i) $E[X - a \,|\, X \geq a] \leq E[Y - a \,|\, Y \geq a]$ for all a (with convention)

(ii) $\Psi_X(a) \leq \Psi_Y(a)$ for all a, for $\Psi_X(a) := E[X \,|\, X \geq a]$ as before.

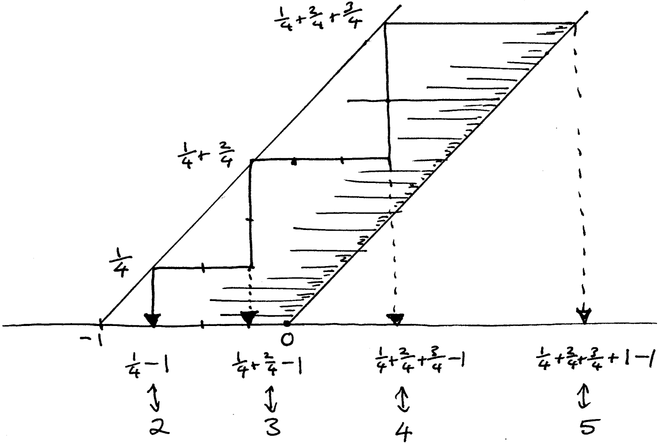
Corollary of the Azéma-Yor embedding:

$$X \leq_{\mathrm{mrl}} Y \text{ and } E(X) = E(Y) \Rightarrow X \leq_{\mathrm{cx}} Y \tag{4}$$

[Shift to $E(X) = E(Y) = 0$, then embed in BM with $T_X \leq T_Y$.]
– van der Vecht (1986), Madan-Yor (2002)
Warning: converse of (4) is false. Indirect argument: if true then T_Y would be an *ultimate time* T for the distribution of Y [Meilijson 1982] meaning:
$B_T \stackrel{d}{=} Y$ and $\forall X$ with $X \leq_{\mathrm{cx}} Y$ $\exists$ stopping $S \leq T$ with $B_S \stackrel{d}{=} X$.
But (Meilijson and van der Vecht, 1980s): *the only ultimate times for BM are the first hitting times of $\{a, b\}$ for some a, b.*

Azéma-Yor embedding of total hazards

Example: Birthday repeat time for $y = 4$ days/year.



Setting: $(\mathcal{F}_t)_{t\geq 0}$, $(T \leq t) \in \mathcal{F}_t$, $P(T < \infty) = 1$

- $A_t - 1(T \leq t)$ is an $(\mathcal{F}_t)$-martingale
- $\to A_\infty - 1$.
- Assume a uniform $[0,1]$ variable U independent of $\mathcal{F}_\infty$.

Theorem

There exists a standard exponential variable ε such that

$$E(\varepsilon \,|\, \mathcal{F}_\infty) = A_\infty \tag{5}$$

$$E[(\varepsilon - A_\infty)^2 \,|\, \mathcal{F}_\infty] = \Delta A_T := A_T - A_{T-} \tag{6}$$

$$E[(\varepsilon - A_\infty)^2] = E[\Delta A_T] = E \sum_s (\Delta A_s)^2 \tag{7}$$

Remarks

- (5) follows from Neveu's inequality that $A_\infty \leq_{\mathrm{mrl}} \varepsilon$.
- (6) involves details of the Azéma-Yor embedding.

Details of the coupling

For each $t > 0$ there is a unique $p = (1 - e^{-t})/t \in (0,1)$ so

$$\xi(t) \stackrel{d}{=} p\, Dist(\varepsilon \,|\, \varepsilon < t) + (1-p) Dist(\varepsilon \,|\, \varepsilon > t) \text{ has}$$

$$E[\xi(t)] = pE(\varepsilon \,|\, \varepsilon < t) + (1-p)E(\varepsilon \,|\, \varepsilon > t) = t$$

Also $Var(\xi(t)) = t$. Explicitly, take U, V, ε' are independent, with U, V uniform$[0,1]$ and $\varepsilon' \stackrel{d}{=} \varepsilon$, and set

$$\xi(t) = tU\, 1(V \le e^{-tU}) + (t + \varepsilon')1(V > e^{-tU}).$$

For A_T a total hazard, take U, V, ε' indpt. of (A_{T-}, A_T). Let

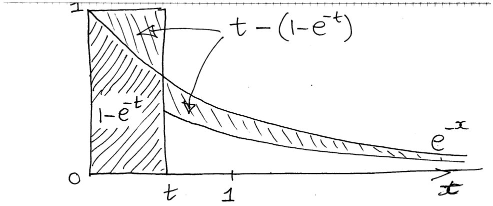
$$\varepsilon = A_{T-} + \xi(A_T - A_{T-}) \text{ so}$$

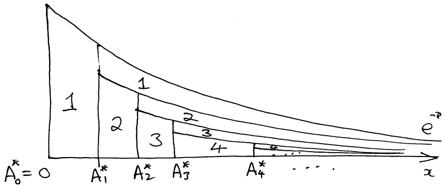
$$E(\varepsilon \,|\, A_{T-}, A_T) = A_T \text{ and } E[(\varepsilon - A_T)^2 \,|\, A_{T-}, A_T) = A_T - A_{T-}$$

Exponential coupling

Example: discrete distribution of X with constant hazards

$$h_n^* = P(X = n \,|\, X \geq n) \text{ and } A_n^* := \sum_{i=1}^{n} h_i*$$





Loose Ends

- Characterize all possible laws of total hazard variables A_T. (Know extemes. Simplex?)
- Can show γ_r/r is $\downarrow$ in MRL as $r\uparrow$. (So Madan-Yor $\Rightarrow$ reverse peacock). $\gamma_1 \stackrel{d}{=} \varepsilon$. Is γ_r/r a total hazard for $r > 1$?
- What about $\gamma_r - r$. Is this $\uparrow$ in MRL?
- Embedding the entire martingale $A_t - 1(T \leq t)$ in BM.
- What about the non-adapted case (martingale derived from a potential)?
- Suppose a stopping time $S \leq T_Y$ where T_Y is the Azéma-Yor time for embedding Y in BM. Does that imply $B_S \leq_{\mathrm{mrl}} Y$?