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CONFIDENCE INTERVALS FOR THE CROSSCOVARIANCE FUNCTION

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1. Introduction. Given a stretch $\{X_1(t), X_2(t)\}$, $t=0,1,\ldots,T-1$, of a stationary bivariate time series, the sample crosscovariance function

$$c_{12}^T(u) = T^{-1} \sum_{0 < t, t+u \leq T-1} \left\{X_1(t+u) - c_1^T\right\} \left\{X_2(t) - c_2^T\right\} \quad (1.1)$$

with

$$c_j^T = T^{-1} \sum_{t=0}^{T-1} X_j(t) \quad (1.2)$$

$j=1,2$ is a useful statistic to employ in looking for association between the component series X_1 and X_2. The exact sampling distribution of the statistic (1.1) is complicated even in simple particular cases. However its distribution has been shown to be asymptotically normal and large sample expressions have been given for its variance, Bartlett (1966), Hannan (1970), Anderson (1971), Brillinger (1975). For example in terms of the second and fourth order spectra of the series $\{X_1, X_2\}$

$$\mathrm{var}\, c_{12}^T(u) \sim 2\pi T^{-1} \left[\int_0^{2\pi} f_{11}(\alpha) f_{22}(\alpha) d\alpha + \int_0^{2\pi} \exp\{i2u\alpha\} f_{12}(\alpha) f_{21}(-\alpha)\, d\alpha + \int_0^{2\pi}\int_0^{2\pi} \exp\{iu(\alpha+\beta)\}\, f_{1212}(\alpha,-\alpha,\beta)\, d\alpha d\beta \right]. \quad (1.3)$$

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In the particular case that the series X_1 and X_2 are independent and that one is a series of independent variates, (white noise), expression (1.3) simplifies to

$$T^{-1}\{\mathrm{var}\ X_1(t)\}\{\mathrm{var}\ X_2(t)\} \qquad (1.4)$$

a result that is often made use of in practice. In this connection Haugh (1976) investigates the effect of approximately transforming the series X_1 and X_2 to white noise and then estimating the crosscovariance function. In the case that the series $\{X_1,X_2\}$ is Gaussian, the fourth order spectrum f_{1212} is identically 0 and expression (1.3) simplifies somewhat. Here Robinson (1977) has shown that a consistent estimate of the variance results from the insertion into (1.3) of consistent estimates of the spectra that appear.

The present paper indicates a direct procedure for constructing approximate confidence intervals for the cross-covariance function, $c_{12}(u)$, of a general bivariate stationary time series $\{X_1,X_2\}$. By taking the series X_2 identical with the series X_1 the procedure leads to the construction of intervals for the autocovariance function $c_{11}(u)$. The procedure is based upon the observation that

$$T^{-1} \sum_{0<t,t+u\leq T-1} X_1(t+u)\ X_2(t) \qquad (1.5)$$

the key variate of (1.1) is for given u, essentially, the mean of the stationary series $X_1(t+u)X_2(t)$, $t=0,\pm 1,\ldots$. Methods are available for basing confidence intervals on such means.

To proceed, it is now necessary to set down some general notation. Given a J vector-valued stationary series $\{X_1(t),\ldots,X_J(t)\}$, $t=0,\pm 1,\ldots$ define

$$c_{j_1\cdots j_k}(u_1,\ldots,u_{k-1}) = \mathrm{cum}\left\{X_{j_1}(t+u_1),\ldots,X_{j_{k-1}}(t+u_{k-1}), X_{j_k}(t)\right\} \qquad (1.6)$$

for $j_1,\ldots,j_k = 1,\ldots,J$ and $k=1,2,\ldots$ in the manner of Brillinger (1975). In particular one has

$$\begin{aligned} c_j &= E\,X_j(t) \\ c_{jk}(u) &= \mathrm{cov}\{X_j(t+u),X_k(t)\} \end{aligned} \qquad (1.7)$$

for $j,k = 1,\ldots,J$. In order to obtain the asymptotic distribution of the statistic under investigation, the following mixing condition will be set down.

<u>Assumption 1</u>. The series $\{X_1(t),\ldots,X_J(t)\}$, $t=0,\pm 1,\ldots$ is stationary and such that

$$\sum_{u_1}\cdots\sum_{u_{k-1}} |c_{j_1\cdots j_k}(u_1,\ldots,u_{k-1})| < \infty \qquad (1.8)$$

for $j_1,\ldots,j_k = 1,\ldots,J$ and $k=2,3,\ldots$.

The vector version of Theorem 2.9.1 of Brillinger (1975) now indicates that for given u, the series

$$X_3(t) = X_1(t+u)X_2(t) \qquad (1.9)$$

$t=0,\pm 1,\ldots$ satisfies Assumption 1 when the series $\{X_1, X_2\}$ does. It has mean

$$c_3 = E\,X_3(t) = c_{12}(u) \qquad (1.10)$$

when c_1 or $c_2 = 0$. The problem of constructing a confidence interval for $c_{12}(u)$ is seen to be directly related to the problem of constructing a confidence interval for the mean of a stationary series.

Further parameters that will be required include the second order spectra

$$f_{jk}(\lambda) = (2\pi)^{-1} \sum_u \exp\{-i\lambda u\}\, c_{jk}(u) \qquad (1.11)$$

for $-\infty<\lambda<\infty$.

2. Confidence Intervals for a Mean. Given a series $X_3(t)$, $t=0,\pm 1,\ldots$ satisfying Assumption 1, it follows from Theorem 4.4.1 of Brillinger (1975) that the sample mean

$$c_3^T = T^{-1} \sum_{t=0}^{T-1} X_3(t) \qquad (2.1)$$

is asymptotically normal with mean c_3 and variance

$$T^{-1} 2\pi f_{33}(0) \qquad (2.2)$$

as $T\to\infty$. Approximate confidence intervals for c_3 may therefore be constructed once an estimate of $f_{33}(0)$ is available.

A variety of estimates of power spectra are available, see for example, Chapter 5 of Brillinger (1975), and the particular one employed in a given situation depends on the details of the situation. To proceed in a concrete fashion, suppose that the estimate constructed is

$$f_{33}^T(0) = L^{-1} \sum_{s=1}^{L} I_{33}^T\left(\frac{2\pi s}{T}\right) \qquad (2.3)$$

where

$$I_{33}^T(\lambda) = (2\pi T)^{-1} \left| \sum_{t=0}^{T-1} X_3(t) \exp\{-i\lambda t\} \right|^2 . \qquad (2.3)$$

It follows from Theorems 4.4.1 and 5.4.3 of Brillinger (1975) that $f_{33}^T(0)$ is asymptotically distributed as

$$f_{33}(0)\chi^2_{2L}/(2L) \quad , \qquad (2.5)$$

independently of c_3^T, as $T\to\infty$ with L fixed. The asymptotic distribution of the variate

$$(c_3^T - c_3) \Big/ \sqrt{T^{-1} 2\pi f_{33}^T(0)} \qquad (2.6)$$

is therefore Student's t with $2L$ degrees of freedom and an approximate $100\beta\%$ confidence interval for c_3 is provided by

$$c_3^T - t_{2L}\left(\frac{1+\beta}{2}\right)\sqrt{T^{-1}2\pi f_{33}^T(0)} < c_3 < c_3^T + t_{2L}\left(\frac{1+\beta}{2}\right)\sqrt{T^{-1}2\pi f_{33}^T(0)} \quad (2.7)$$

where for Student's t on $2L$ degrees of freedom

$$\text{Prob}\{t < t_{2L}(\gamma)\} = \gamma \ .$$

In practice one will take L as large as one feels one can, consistent with $f_{33}(\lambda)$ essentially constant on $0 \leq \lambda \leq 2\pi L/T$. If one takes $L = T$ in (2.3), then

$$f_{33}^T(0) = (2\pi T)^{-1} \sum_{t=0}^{T-1} \left\{X_3(t) - c_3^T\right\}^2$$

and expression (2.6) is essentially the usual t pivotal quantity employed in constructing a confidence interval for the mean of a normal sample.

It is worth mentioning that Grenander (1950) showed, provided $f_{33}(0) \neq o$, the simple mean (2.1) to be an asymptotically efficient estimate of c_3 within the class of linear unbiased estimates. Hence, nothing is to be gained asymptotically, by differential weighting of the individual observations. In the case that T, the number of data points available, is highly composite a fast Fourier transform algorithm might be

profitably employed in the computation of (2.3), (2.4) and the choice of L, Markel (1971).

In the case that $\underset{\sim}{X}(t)$, $t = 0, \pm 1, \ldots$ is a vector-valued series satisfying Assumption 1 with mean $\underset{\sim}{c}$ and spectral density matrix $\underset{\sim}{f}(\lambda)$, the sample mean

$$\underset{\sim}{c}^T = T^{-1} \sum_{t=0}^{T-1} \underset{\sim}{X}(t) \quad (2.8)$$

is asymptotically normal with mean $\underset{\sim}{c}$ and covariance $T^{-1}2\pi \underset{\sim}{f}(0)$ as $T \to \infty$, see Theorem 4.4.1 in Brillinger (1975). Consider now the problem of setting confidence limits for some function, $g(\underset{\sim}{c})$, of the mean $\underset{\sim}{c}$. Suppose that g has vector of first derivatives, $\underset{\sim}{\gamma}_c$, at $\underset{\sim}{c}$ continuous in a neighborhood of $\underset{\sim}{c}$. It follows from Mann and Wald (1943) that

$$g(\underset{\sim}{c}^T) = g(\underset{\sim}{c}) + \underset{\sim}{\gamma}_c'(\underset{\sim}{c}^T - \underset{\sim}{c}) + o_p(T^{-1/2}) \quad (2.9)$$

and that $g(\underset{\sim}{c}^T)$ is asymptotically normal with mean $g(\underset{\sim}{c})$ and variance

$$T^{-1}2\pi \underset{\sim}{\gamma}_c' \underset{\sim}{f}(0) \underset{\sim}{\gamma}_c \ . \quad (2.10)$$

Approximate confidence intervals for $g(\underset{\sim}{c})$ may hence be based on the quantity

$$\{g(\underset{\sim}{c}^T) - g(\underset{\sim}{c})\} \Big/ \sqrt{T^{-1}2\pi \underset{\sim}{\gamma}_{c^T}' \underset{\sim}{f}^T(0) \underset{\sim}{\gamma}_{c^T}} \quad (2.11)$$

where the denominator here involves an estimate of the variance (2.10). In practice this estimate may be formed by first evaluating the estimate $\underset{\sim}{f}^T(0)$ of the matrix $\underset{\sim}{f}(0)$ and then the quadratic form. Alternatively the real-valued series

$$X_3(t) = \underset{\sim}{\gamma}_C^{\prime T} \underset{\sim}{X}(t) \tag{2.12}$$

might first be formed and then the statistic (2.3) constructed. This remark follows from the observation based on (2.9) that $g(\underset{\sim}{c}^T)$ is approximately linear in a mean.

3. The Case of Interest. To begin, suppose that either mean c_1 or c_2 is 0, then under Assumption 1

$$c_{12}^T(u) = T^{-1} \sum_{0 \le t, t+u < T-1} \left\{X_1(t+u) - c_1^T\right\}\left\{X_2(t) - c_2^T\right\}$$

$$= T^{-1} \sum_{t=0}^{T-1} X_1(t+u)X_2(t) + o_p(T^{-1/2})$$

as $T \to \infty$. It follows that the asymptotic distribution of $c_{12}^T(u)$ is the same as that of the sample mean of the series $X_3(t) = X_1(t+u)X_2(t)$, $t=0,\pm1,\ldots$ for given u. The discussion of the preceding section now suggests employing the pivotal quantity

$$\left\{c_{12}^T(u) - c_{12}(u)\right\} \Big/ \sqrt{T^{-1} 2\pi f_{33}^T(0)} \tag{3.1}$$

with $f_{33}^T(0)$ given by (2.3) and suggests approximating the distribution of (3.1) by Student's t with $2L$ degrees of freedom.

It is interesting to note that the term within $[\cdot]$ of expression (1.3) is $f_{33}(0)$ for the series $X_3(t) = X_1(t+u) \times X_2(t)$ here. It is also worth remarking that following Grenander's result mentioned in the previous section, no asymptotic gain will result here from a differential weighting of the terms $X_1(t+u)X_2(t)$ used in forming the estimate of $c_{12}(u)$.

In the case of interest where $c_1, c_2 \neq 0$, the series X_3 will be defined by

$$X_3(t) = \left\{X_1(t+u) - c_1^T\right\}\left\{X_2(t) - c_2^T\right\} \tag{3.2}$$

and the estimate $f_{33}^T(0)$ based on its values. Under Assumption I, this change will have no asymptotic effect.

In summary, an approximate $100\beta\%$ confidence interval for the parameter $c_{12}(u)$ is provided by

$$c_{12}^T(u) - t_{2L}\left(\frac{1+\beta}{2}\right)\sqrt{T^{-1}2\pi f_{33}^T(0)} < c_{12}(u) < c_{12}^T(u) + t_{2L}\left(\frac{1+\beta}{2}\right)\sqrt{T^{-1}2\pi f_{33}^T(0)} \tag{3.3}$$

where $f_{33}^T(0)$ is given by (2.3) and (3.2).

The interval (3.3) is also appropriate for the auto-covariance $c_{11}(u)$ of a single series. This case simply corresponds to the previous case with the series X_2 identical to the series X_1.

4. Alternate Statistics. Statistics other than the sample crosscovariance function are also in common use to examine two stationary time series for mutual independence. The one in greatest use is perhaps the sample crosscorrelation function

$$r_{12}^T(u) = c_{12}^T(u) \Big/ \sqrt{c_{11}^T(0) c_{22}^T(0)} \qquad (4.1)$$

estimating the crosscorrelation function $r_{12}(u) = c_{12}(u) / \sqrt{c_{11}(0) c_{22}(0)}$. This last parameter takes on values in the interval $[-1,1]$ and is 0 if the series are independent. Defining the series

$$\begin{aligned} X_4(t) &= \left\{X_1(t+u) - c_1^T\right\} \left\{X_2(t) - c_2^T\right\} \\ X_5(t) &= \left\{X_1(t) - c_1^T\right\}^2 \\ X_6(t) &= \left\{X_2(t) - c_2^T\right\}^2 \end{aligned} \qquad (4.2)$$

$t=0,\pm 1,\ldots$ the variate $r_{12}^T(u)$ is seen to be essentially a function of sample means

$$\begin{aligned} r_{12}^T(u) &\sim c_4^T \Big/ \sqrt{c_5^T c_6^T} \\ &= c_4 / \sqrt{c_5 c_6} + \left(c_4^T - c_4\right) / \sqrt{c_5 c_6} - c_4 \left(c_5^T - c_5\right) / \left(2 c_5 \sqrt{c_5 c_6}\right) \\ &\quad - c_4 \left(c_6^T - c_6\right) / \left(2 c_6 \sqrt{c_5 c_6}\right) + o_p(T^{-1/2}) \end{aligned} \qquad (4.3)$$

in the manner of expression (2.9) . Following the discussion at the end of Section 2, an approximate $100\beta\%$ confidence interval is provided by

$$r_{12}^T(u) - t_{2L}\left(\frac{1+\beta}{2}\right) \sqrt{T^{-1} 2\pi f_{33}^T(0)} < r_{12}(u) < r_{12}^T(u) + t_{21}\left(\frac{1+\beta}{2}\right) \sqrt{T^{-1} 2\pi f_{33}^T(0)} \qquad (4.4)$$

where $f_{33}^T(0)$ is given by (2.3) and

$$X_3(t) = r_{12}^T(u) \left\{ X_4(t)/c_{12}^T(u) - X_5(t)/(2 c_{11}^T(0)) - X_6(t)/(2 c_{22}^T(0)) \right\} \qquad (4.5)$$

in terms of the series of (4.2) . In practice on occasion it will prove simpler to estimate the power and cross spectra of the series of (4.2) and then to evaluate the quadratic form of (2.11).

A further means of examining a pair of stationary time series for some mutual association is based on the coherence function

$$|R_{12}(\lambda)|^2 = |f_{12}(\lambda)|^2/(f_{11}(\lambda)f_{22}(\lambda)). \quad (4.6)$$

This parameter takes on values in the interval $[0,1]$ and vanishes when the series X_1 and X_2 are statistically independent. It may be estimated by a statistic of the form

$$|R_{12}^T(\lambda)|^2 = |f_{12}^T(\lambda)|^2/(f_{11}^T(\lambda)f_{22}^T(\lambda)) \quad (4.7)$$

where $f_{jk}^T(\lambda)$ is an estimate of $f_{jk}(\lambda)$. Provided T is reasonably large such estimates generally prove useful. Approximations are available for the distributions and for setting confidence limits, see Chapter 8 in Brillinger (1975). The computations required here are noticeably less then those in the cases of $c_{12}^T(u)$ and $r_{12}^T(u)$; however the latter statistics are undoubtedly most useful in cases where T is not overly large.

5. Concluding Remarks. The asymptotic results made use of in this paper, were derived under the assumption that the series under consideration satisfied the mixing Assumption I. In fact, the results might have been derived under a number of other mixing assumptions. For example, Hannan (1976) demonstrates that the sample autocovariance function is asymptoticaally normal under an alternate such condition.

The multiplier T^{-1} was employed in the definition of the estimate $c_{12}^T(u)$ in expression (1.1). In some circumstances one chooses to employ $(T-|u|)^{-1}$. This change has little effect for $|u|$ of moderate size relative to T. In the other cases one might choose to replace T by $T-|u|$ in expression (3.3).

The asymptotic variance (1.3) is not generally constant as a function of u. In practice it is an advantage to have it approximately constant if one is computing the statistic for a number of lags u. This will occur if the spectra f_{12}, f_{1212} are approximately constant. Hence, it will generally be advantageous to prewhiten the relationship between the series where possible. For example, in some circumstances it might be acceptable to form the sample cross covariance function of the series X_1 with the residuals of a regression of the series X_2 on a number of lagged values of the series X_1.

The distribution of the statistic (1.1) was considered in this paper for the case of $\{X_1,X_2\}$ a bivariate stationary series. On occasion its distribution is of interest for the case of X_1 a stationary series, but X_2 a fixed sequence. Hannan (1973) and Brillinger (1975), Sections 5.11, 6.12 consider this situation for the case of the series X_2 satisfying a generalized harmonic analysis condition.

REFERENCES

ANDERSON, T.W. (1971). The Statistical Analysis of Time Series. New York, J. Wiley.

BARTLETT, M.S. (1966). An Introduction to Stochastic Processes. Cambridge, Cambridge University Press.

BRILLINGER, D.R. (1975). Time Series: Data Analysis and Theory. New York, Holt, Rinehart and Winston.

GRENANDER, U. (1950), Stochastic processes and statistical inference. Ark. Mat. 1, 195-277.

HANNAN, E.J. (1970). Multiple Time Series. New York, J.Wiley.

HANNAN, E.J. (1973). Central limit theorems for time series regression. Zeit. Wahrschein. 26, 157-170.

HANNAN, E.J. (1976). The asymptotic distribution of serial covariances. Ann. Stat. 4, 396-399.

HAUGH, L.D. (1976). Checking the independence of two covariance stationary time series. J. Amer. Statist. Assoc. 71, 378-385.

MANN, H.B. and WALD, A. (1943). On stochastic limit and order relationships. Ann. Math. Stat. 14, 217-226.

MARKEL, J.D. (1971). FFT pruning. IEEE Trans. Audio Electro. AU-19, 305-311.

ROBINSON, P.M. (1978). Estimating variances and covariances of sample autocorrelations and autocovariances. Australian J. Stat. 18.

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ABSTRACT

Approximate confidence intervals are constructed for the auto and cross-covariance functions and for the auto and cross-correlation functions of a bivariate stationary time series satisfying a particular asymptotic independence (or mixing) condition. The construction proceeds via the estimation of the power spectrum at frequency 0 of a related series, quadratic in the original series.