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SOME ASYMPTOTICS OF FINITE FOURIER TRANSFORMS OF A STATIONARY p-ADIC PROCESS

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The result that in some circumstances finite Fourier transforms are approximately normally distributed, has proved useful for suggesting a number of probabilistic results and statistical procedures in the case of ordinary time series. There has been some study for the case of a stationary random function defined on a locally compact Abelian group. In this paper the particular case of stationary random functions of p-adic numbers is studied in some preliminary detail. Some interesting simplifications occur. Certain sequences of Fourier transforms are found to be asymptotically normal. An empirical spectral process, with time parameter in a function space, is found to converge in distribution. Another work studying process defined over a particular group is Taniguchi, Zhao, Krishnaiah and Bai (1989).

1. INTRODUCTION

These days p-adic numbers are finding application in the fields of error-free computation, see Gregory and Krishnamurthy (1984) and to quantum mechanics and quantum field mechanics see Vladimirov (1988), Vladimirov and Volovich (1989). A motivation for considering p-adic space provided in the former is that "In the theory of superstrings... which appeals to fantastically small distances of the order of 10^{-33} cm., there is no reason to assume that the ordinary representations of space-time are applicable". P-adic numbers are also appearing in stochastic circumstances, see Madrecki (1985), Michailov (1986), Evans (1988a, b), (1989) and Vladimirov and Volovich (1989), Section 5. In this paper central limit theorems are developed for finite Fourier transforms and for a family of quadratic statistics based on a real-valued stochastic process $Y(t)$, with $t \in Q_p$, the field of p-adic numbers.

For the case of a discrete time stationary mixing 0 mean stationary time series $Y(t)$ with $t \in Z$, as n tends to ∞ the finite Fourier transform

$$d^n(\lambda) = \sum_0^{n-1} \exp\{-i\lambda t\} Y(t) \tag{1.1}$$

$\lambda \in (0, 2\pi]$ is asymptotically normal with mean 0 and variance $2\pi n f_2(\lambda)$, $f_2(\cdot)$ being the power spectrum of Y. A variety of references related to this result may be found in Brillinger (1982). Among the uses of the result are: (i) confidence intervals for the mean of a stationary process, (ii) power spectrum estimates, (iii) higher-order spectrum estimates, (iv) spectral measure estimates and (v) Gaussian estimates of a finite dimensional parameter. In this paper a few analogs are developed for the p-adic case.

Before studying random p-adic functions, some basic details of the p-adic numbers themselves must be set down.

2. The P-Adic Numbers

2.1. The Field Q_p

It is usual to carry out analysis of functions of real numbers or of complex numbers. These domains are both locally compact topological fields with many special properties and are distinguished by being connected. There is a disconnected locally compact field that is currently enjoying concentrated study, the field Q_p of p-adic numbers. Here p is any prime number. There are several methods to introduce the field of p-adic numbers and the corresponding ring of p-adic integers.

In abstract fashion one can proceed as follows. Let p be a prime. Let Z be the ring of integers and Q the field of rationals. For $a, b \neq 0 \in Z$ define the norm

$$|a/b|_p = p^{n-m}$$

with m the highest power of p dividing a and n the highest power of p dividing b. Finish the definition via $|0|_p = 0$. The (topological) field of p-adic numbers, Q_p, may now be defined as the completion of Q in the metric defined by the norm $|\cdot|_p$. The operations of $+$, $-$, $\times$, $/$ carry over from Q. This space is fundamental because a theorem of Ostrowski indicates that any norm on Q is either the usual Euclidean norm or $|\cdot|_p$ for some p, see Koblitz (1980). The ring of p-adic integers, Z_p, is given by the elements of Q_p satisfying $|t|_p \leqslant 1$.

The p-adic numbers may be introduced in more concrete fashion as follows. They are symbolic expressions of the form

$$t = t_m p^m + t_{m+1} p^{m+1} + \dots \tag{2.1}$$

with $t_i \in \{0, 1, \dots, p-1\}$, and m any integer, positive or negative. If $t_m \neq 0$ then the norm of this p-adic number t is defined to be $|t|_p = p^{-m}$.

For carrying out p-adic arithmetic it is convenient to represent t as

$$p^m\{t_m, t_m + t_{m+1}p, t_m + t_{m+1}p + t_{m+2}p^2, \dots\}$$

Addition or multiplication of two p-adic numbers is then carried out coordinatewise (mod p), adding 0's on the left as appropriate. See for example Borevich and Shafarevich (1966). The series of partial sums in (2.1), which is a series of rational numbers, converges to t in Q_p.

By $\langle t\rangle_p$, the fractional part of the p-adic number (2.1), is meant

$$\langle t\rangle_p = \sum_{j<0} t_j p^j$$

evaluated as a real number lying in the interval (0, 1). (To simplify notation, it will generally be written $\langle t\rangle$ in what follows.) One has $\langle t\rangle_p \leqslant p\,|t|_p$.

It is perhaps worth stating that the p-adics are not extensions of the dyadics. For the dyadics there is no tying together of the coordinates.

2.2. Integration and Fourier analysis on Q_P

Q_p is a locally compact additive Abelian group. An invariant (Haar) measure therefore exists for Q_p. It has the properties $d(t+a)=dt$ and $d(at)=|a|_p\,dt$. The Haar measure of the Borel set A of Q_p will be denoted $\mu(A)$. The measure will be normalized by $\mu(Z_p)=1$. The form of Haar measure is given in Hewitt and Ross (1963), pages 202–203. To illustrate it, consider first an integral over Z_p. Since $\mu(Z_p)=1$ it may be considered an expected value. In fact writing $t=t_0+t_1p+t_2p^2+\cdots$ and $f(t)=g(t_0, t_1, t_2, \ldots)$ and taking $(T_0, T_1, T_2, \ldots\}$ to be a sequence of i.i.d. random variables T on the sample space $\{0, 1, \ldots, p-1\}$ with equal probability of selection one has

$$\int_{Z_p} f(t)\,dt = E\{g(T_0, T_1, T_2, \ldots)\}$$

Supposing the integral of $f(.)$ to be given by

$$\int_{Q_p} f(t)\,dt = \lim_{n\to\infty} \int_{|t|_p\leqslant p^n} f(t)\,dt$$

the integral on the right may be represented as

$$\int_{|t|_p\leqslant p^n} f(t)\,dt = p^n \int_{|s|_p\leqslant 1} f(p^{-n}s)\,ds$$

reducing to an integral of the previous form.

Because of the group nature of Q_p characters, $\lambda(t)$, providing a Fourier analysis exist. These are the unit modulus, complex-valued, multiplicative, continuous functions on the group. They have the form, see Hewitt and Ross (1963), pages 400–402, or Gelfand *et al.* (1969)

$$\lambda(t) = \exp(2\pi i\langle\lambda t\rangle)$$

for $\lambda \in Q_p$ with $\langle \lambda t \rangle$, as before, denoting the fractional part of the p-adic number λt.

A variety of p-adic Fourier transform pairs have been determined, see for example Taibelson (1975) or Vladimirov (1988). It is notable that discs Fourier transform into discs, see (2.2) and (2.3) below.

For an integer n let $U_n = p^{-n} Z_p = \{t : |t|_p \leqslant p^n\}$. Let

$$D^n(\lambda) = \int_{U_n} \exp(-2\pi i \langle \lambda t \rangle)\, dt \tag{2.2}$$

This Fourier transform may be evaluated and found to be

$$\begin{aligned} D^n(\lambda) &= p^n \quad \text{for } |\lambda|_p \leqslant p^{-n} \\ &= 0 \text{ otherwise} \end{aligned} \tag{2.3}$$

A theory of generalized functions of p-adic variables has been developed, see Gelfand *et al.* (1969), Taibelson (1975), Vladimirov (1988). This theory proves convenient in setting down succinctly a variety of expressions and in carrying through Fourier analysis.

3. Random Functions of P-adic Numbers

3.1. P-adic processes

Since Q_p is a complete separable metric space, the stochastic process $Y(t, \omega)$ for $t \in Q_p$ and $\omega \in \Omega$, (Ω, A, P) a probability space, is well-defined as a map from $Q_p \times \Omega$ to R. The work of this paper will be further simplified by assuming that the process $Y(\cdot)$ is of second order, that is

$$E\{|Y(t)|^2\} < \infty$$

for all $t \in Q_p$, see eg. Grenander (1981). It will be further assumed that the realizations are real-valued and continuous in mean square that is

$$\lim_{s \to t} E\{|Y(t) - Y(s)|^2\} = 0$$

The meaning of this last is that for given $\epsilon > 0$ and t there exists $N = N(t, \epsilon)$ such that $E\{|Y(t) - Y(s)|^2\} < \epsilon$ for $|t - s|_p < p^N$. In the case that s, t are rational, this last means that the numerator of $t - s$ is divisible by a (high) power of p.

Gaussian processes are those all of whose finite dimensional distributions are Gaussian. In particular the standard Wiener process, with parameter set Q_p, is an additive process on the ring of Borel sets of Q_p with the properties: (i) $W(A)$ is Gaussian, (ii) $E\{W(A)\} = 0$ and (iii) $E\{W(A)W(B)\} = |A \cap B|$. A, B being Borel sets. Its "derivative" is standard white noise with independent Gaussian values at each point.

The Wiener process is useful in the construction of other processes, e.g. the Gaussian linear process $Y(t) = \int a(t-u)W(du)$. Evans (1988a), pages 414–415, employs a countable number of standard independent normal variates to construct linearly a Gaussian process with covariance function

$$\operatorname{cov}\{Y(s), Y(t)\} = \max\{|s|_p^\alpha, |t|_p^\alpha\} - |s-t|_p^\alpha$$

for any $\alpha > 0$. This, perhaps, is the p-adic analog of a fractional Brownian field with covariance function $\|s\|^\alpha + \|t\|^\alpha - \|s-t\|^\alpha$.

3.2. The wide sense stationary case

The 0 mean p-adic process $Y(t)$ will be said to be wide sense stationary if

$$E\{Y(t+u)Y(t)\} = \operatorname{cov}\{Y(t+u), Y(t)\} = c_2(u)$$

for all $t, u \in Q_p$. Following the remarks of the previous section, $c_2(u)$ will be assumed continuous at $u = 0$. Under these conditions the autocovariance function and the process itself have spectral representations

$$c_2(u) = \int_{Q_p} \exp(2\pi i\langle u\lambda\rangle)F_2(d\lambda) \tag{3.1}$$

with F_2 a finite measure on Q_p and

$$Y(t) = \int_{Q_p} \exp(2\pi i\langle t\lambda\rangle)Z_Y(d\lambda) \tag{3.2}$$

where Z_Y is a spectral process with the properties $E\{Z_Y(d\lambda)\} = 0$ and

$$E\{Z_Y(d\lambda)Z_Y(d\mu)\} = \delta(\lambda + \mu)F_2(d\lambda)\,d\mu$$

see e.g. Kampé de Fériet (1949) or Neumann (1965). Here $\delta(\lambda)$ is the p-adic Dirac delta function. The process Z_Y will be complex-valued, but since Y is real-valued it will satisfy $\overline{Z_Y(d\lambda)} = Z_Y(-d\lambda)$. (The overbar denotes "complex conjugate".) Given a stationary process, $Y(.)$, other stationary processes may now be constructed by linear time-invariant filtering, e.g. $\int \exp(2\pi i\langle t\lambda\rangle)A(\lambda)Z_Y(d\lambda)$ with $\int |A(\lambda)|^2F_2(d\lambda) < \infty$.

To make things clearer some examples are presented.

Example 3.1. The following Fourier transform pair may be found in Taibelson (1975), page 22

$$f(\lambda) = |\lambda|_p^\alpha \quad \text{for } |\lambda|_p \leqslant 1$$
$$= 0 \text{ otherwise}$$

and

$$c(u) = \int_{Q_p} \exp(2\pi i\langle u\lambda\rangle)f(\lambda)\,d\lambda$$

$$= \frac{1-p^{-1}}{1-p^{-(1+\alpha)}} \text{ for } |u|_p \leqslant 1$$

$$= \frac{1-p^{-1}}{1-p^{-(1+\alpha)}} |u|_p^{(1+\alpha)} \text{ for } |u|_p > 1$$

Since $f(.)$ is non-negative and integrable for $\alpha < -1$ this pair provides an autocovariance and corresponding power spectrum. A Gaussian process with these parameters may be represented

$$Y(t) = \int_{|\lambda|_p \leqslant 1} \exp(2\pi i \langle t\lambda \rangle) |\lambda|_p^{\alpha/2} Z(d\lambda) \tag{3.3}$$

with $Z(d\lambda)$ a complex Wiener process. The unit correlation of the process for $|u|_p \leqslant 1$ may be puzzling at first sight, but examination of (3.3) shows that the process $Y(t)$ is periodic, that is $Y(t+a) = Y(t)$ for $|a|_p \leqslant 1$.

EXAMPLE 3.2. P-adic valued random variables have been considered in Madrecki (1985) and Evans (1988c). Let ω denote a realization of a p-adic random variable with probability element $F(d\omega)$ on Q_p. Let ϕ denote a random variable uniform on $(0, 2\pi]$. Let a be a real constant. Set

$$Y(t) = a \cos(2\pi \langle t\omega \rangle + \phi)$$

for $t \in Q_p$. Then the autocovariance function of the process, $Y(.)$, is given by

$$c_2(u) = \frac{a^2}{2} \int_{Q_p} \exp(2\pi i \langle u\omega \rangle)[F(d\omega) + F(-d\omega)]/2$$

And so one sees one can achieve a process with (second-order) spectral measure proportional to the probability measure of any symmetric p-adic random variable. By adding many independent realizations of this process, one can achieve an approximately Gaussian process with general spectral measure.

P-adic random variables may be constructed from ordinary ones in the manner that Haar measure was constructed above. Let $\{M, T_0, T_1, \ldots\}$ be a random variate on the sample space $Z \times P \times P \times \ldots$ where $P = \{0, 1, \ldots, p-1\}$. Then $p^M(T_0 + T_1 p + T_2 p^2 + \ldots)$ is such a variate.

EXAMPLE 3.3 Suppose $Y(.)$ is Gaussian with $F_2(d\lambda) = f_2(\lambda) d\lambda$. Then $\sqrt{f_2(.)}$ is square integrable. Let its Fourier transform be $a(.)$. Then $Y(.)$ has a linear process representation

$$Y(t) = \int a(t-u) W(du)$$

with $W(.)$ a standard Wiener process.

3.3. Full Stationarity and Mixing

As is implicit in the last example, in the case that the process $Y(.)$ is

Gaussian, the first and second moments will suffice to define it. In the general case more will be needed. In the approach of the present paper, the additional detail will be provided by higher-order moments and cumulants. To begin, it will be assumed that the spectral measure $F_2(d\lambda)$ is absolutely continuous with respect to the Haar measure, $F_2(d\lambda) = f_2(\lambda)d\lambda$, $f_2(\lambda)$ the power spectrum at frequency λ. In this case, because $f_2(.)$ is non-negative, $c_2(.)$ is absolutely integrable and one has the inverse relationship

$$f_2(\lambda) = \int_{Q_p} \exp\left(-2\pi i\langle\lambda u\rangle\right)c_2(u)\, du$$

The principal requirement for developing limit theorems will be:

Assumption 3.1 The process $Y(t)$, $t \in Q_p$ is continuous in mean-square, has zero mean and is stationary with moments of all orders existing and cumulant functions

$$c_k(u_1, \ldots, u_{k-1}) = \text{cum}\,\{Y(t+u_1), \ldots, Y(t+u_{k-1}), Y(t)\}$$

absolutely integrable for $k = 2, 3, \ldots$.

Here "cum" denotes the joint cumulant of the variates indicated. Properties of these quantities for random processes are set down in Brillinger (1975), for example.

The cumulant spectra of the process are now defined as

$$f_k(\lambda_1, \ldots, \lambda_{k-1})$$
$$= \int \cdots \int \exp\left(-2\pi i\langle\lambda_1 u_1 + \ldots + \lambda_{k-1}u_{k-1}\rangle\right)c_k(u_1, \ldots, u_{k-1})du_1 \ldots du_{k-1}$$

with the further interpretation

$$\text{cum}\,\{Z_Y(d\lambda_1), \ldots, Z_Y(d\lambda_k)\} = \delta(\lambda_1 + \ldots + \lambda_k)f_k(\lambda_1, \ldots, \lambda_{k-1})d\lambda_1 \ldots d\lambda_k$$

That these spectra are concentrated on hyperplanes results from the assumed stationarity of the process.

4. A Central Limit Theorem

4.1. The Finite Fourier Transform

Study now turns to a p-adic form of the finite Fourier transform (1.1). An analog in the p-adic case is provided by

$$d^n(\lambda) = \int_{U_n} \exp\left(-2\pi i\langle\lambda t\rangle\right)Y(t)\, dt \qquad (4.1)$$

where, for example, $U_n = p^{-n}Z_p = \{t : |t|_p \leqslant p^n\}$. The Haar measure of U_n is $\mu(U_n) = p^n$ and tends to infinity as $n \to \infty$.

The sequence of variates, corresponding to $\lambda = 0$ in (4.1),

$$S_n = \int_{U_n} Y(t)\, dt$$

$n = 0, 1, 2, \ldots$ provides an analog of the sample sums typically considered in central limit theory. It will be seen below that S_n and $d^n(\lambda)$ are asymptotically normal as $n \to \infty$.

4.2. The Cumulants

There are surprisingly simple expressions for the joint cumulants of the variates $d^n(\lambda)$.

Lemma 4.1 Under Asumption 3.1

$$\begin{aligned} &cum\{d^n(\lambda_1), \ldots d^n(\lambda_k)\} \\ &= D^n(\lambda_1 + \ldots + \lambda_k) \int_{V_n} \exp(-2\pi i\langle\lambda_1 u_1 + \ldots + \lambda_{k-1}\rangle) \\ &\quad \times c_{k-1}(u_1, \ldots, u_{k-1}) du_1 \ldots du_{k-1} \end{aligned} \tag{4.2}$$

where $V_n = \{|u_1|_p, \ldots, |u_{k-1}|_p \leqslant p^n\}$.

The proof of this and the other results may be found in the Proofs section.

The cumulant is seen to be zero for $|\lambda_1 + \ldots + \lambda_k|_p > p^{-n}$. For large n and $|\lambda_1 + \ldots + \lambda_k|_p \leqslant p^{-n}$ it will be approximately $p^n f_k(\lambda_1, \ldots, \lambda_{k-1})$. In particular

$$var\{d^n(\lambda)\} = p^n \int_{|u|_p \leqslant p^n} \exp(-2\pi i\langle\lambda u\rangle) c_2(u) du \approx p^n f_2(\lambda)$$

for large n

4.3. Asymptotic Normality

Lemma 4.1 may be used to develop the following theorems.

THEOREM 4.1. *Under Assumption* 3.1 *and non-vanishing of* $f_2(\cdot)$ *at* $\lambda=0$, *the "sample sum"* S_n *is asymptotically normal with mean* 0 *and variance* $p^n f_2(0)$ as $n \to \infty$.

Supposing the process has non-zero mean, c_1 this result may be used to develop an approximate confidence interval for c_1 based on the sample mean.

THEOREM 4.2. *Under Assumption* 3.1 *and non-vanishing of* $f_2(\lambda_j)$, *for distinct* $\lambda_j \neq 0$ *the* finite *Fourier transforms* $d^n(\lambda_1), \ldots, d^n(\lambda_J)$ *are asymptotically independent complex normals with mean* 0 *and variances* $p^n f_2(\lambda_j)$ *respectively as* $n \to \infty$.

Morretin (1980) developed a result such as this for a general locally compact noncompact abelian group.

4.4. Quadratic Statistics

An important statistic in the study of stationary processes is given by the periodogram

$$I^n(\lambda) = p^{-n}|d^n(\lambda)|^2$$

It has, following Lemma 4.1, expected value

$$\int_{|u|_p \leq p^n} \exp(-2\pi i \langle \lambda u \rangle) c_2(u) du$$

This expected value is seen to tend to $f_2(\lambda)$ as $n \to \infty$.

A useful family of quadratic statistics based on the periodogram is provided by

$$J^n(A) = \int A(\lambda) I^n(\lambda) d\lambda = p^{-n} \int_{W_n} a(s-t) Y(s) Y(t) \, ds dt = \tilde{J}^n(a)$$

where A is the Fourier transform of a and $W_n = \{s, t : |s|_p, |t|_p \leq p^n\}$.

In particular one might be interested in estimating the spectral measure $F_2(\cdot)$ and consider for example

$$\hat{F}_2(S) = \int_S I^n(\lambda) \, d\lambda$$

for a compact set S. This estimate was proposed in Neumann (1965).

For large n the expected value of $J^n(A)$ will be approximately

$$J(A) = \int A(\lambda) f_2(\lambda) \, d\lambda = \int a(u) c_2(u) \, du = \tilde{J}(a).$$

Further, by elementary computations

$$p^n \operatorname{cov}\{\tilde{J}^n(a), \tilde{J}^n(b)\} \approx 2 \iiint a(u) b(v) c_2(w+u-v) c_2(w) \, du \, dv \, dw$$

$$+ \iiint a(u) b(v) c_4(u+w, v, w) \, du \, dv \, dw \tag{4.3}$$

Under regularity conditions this last may be written

$$2 \int A(\lambda) B(\lambda) f_2(\lambda)^2 \, d\lambda + \iint A(\lambda) B(\lambda) f_4(\lambda, \mu, -\lambda) \, d\lambda \, d\mu.$$

Theorem 4.3. *Under assumption* 3.1 *and assuming the* $a(\cdot)$ *absolutely integrable, finite collections of quadratic statistics* $\tilde{J}^n(a)$ *are asymptotically normal with mean* $E\{\tilde{J}^n(a)\}$ *and covariance given by* (4.3).

Under the given assumptions, $E\{\tilde{J}^n(a)\} \to \tilde{J}(a)$ as $n \to \infty$ and the speed of convergence may be controlled by introducing further regularity conditions.

5. A Functional Central Limit Theorem for Quadratic Statistics

Dahlhaus (1988) lists some of the applications of the asymptotic normality of quadratic statistics in the case of ordinary time series and goes on to indicate more resulting from convergence in distribution of $J^n(A) - J(A)$ as a stochastic process with time parameter A. These include estimating a finite dimensional parameter by maximizing an approximate Gaussian likelihood. A second reference is Doukhan and Leon (1989). Now some results are indicated for the time parameter in a locally compact abelian group, such as Q_p.

Henceforth in the paper the process $Y(\cdot)$ will be assumed Gaussian to simplify the development. Define a process

$$X^n(a) = p^{n/2}[\tilde{J}^n(a) - E\{\tilde{J}^n(a)\}]$$

for the time parameter $a \in \Xi$, a space of functions. Suppose,

Assumption 5.1. Ξ is a totally bounded subset of $L_1(Q_p)$.

The L_1 norm of $a(\cdot)$ will be denoted $\rho(a)$. A condition for total boundedness will be indicated below.

Concern will be for the convergence of the sequence of processes $\{X^n(a), a \in \Xi\}$ to a Gaussian process, $X(\cdot)$, with mean 0 and covariance function

$$\operatorname{cov}\{X(a), X(b)\} = 2 \iiint a(u)b(v)c_2(w + u - v)c_2(w)\, du\, dv\, dw \quad (5.1)$$

Special interest derives from the general nature of the time parameter, a. Because the process $Y(\cdot)$ is now assumed Gaussian, the $c_4(\cdot)$ term drops out of (4.3). The technology of Pollard (1984, 89) will be employed.

Assumption 5.2. The process $\{Y(t), t \in Q_p\}$ is zero mean stationary Gaussian with absolutely integrable autocovariance function and sample paths bounded for t in U_n, for n sufficiently large.

The results of Evans (88a) may be invoked to obtain conditions under which sample paths are continuous for the process segments $\{Y(t), t \in U_n\}$ and so boundedness in sup norm occurs.

For given $\delta > 0$ one introduces the covering number of Ξ, namely $N(\delta) = N(\delta, \rho, \Xi)$, equal to

$$\inf\{m : \text{there exist functions } a_1, \ldots, a_m \in \Xi \text{ with } \inf_i \rho(a - a_i) \leqslant \delta \text{ for all } a \in \Xi\}$$

Now for $\gamma > 0$, that can be arbitrarily small, set

$$\Psi(\eta) = 2 \exp\{-\eta^2/[4\rho(c_2)(c_2(0) + \gamma\eta)]\} \quad (5.2)$$

If $N(\cdot, \rho, \Xi)$ is finite, then Ξ will be totally bounded, see Pollard (1989), Section 3. Write $H(u)$ for $\Psi^{-1}[N(u)^2/u]$. It increases as u decreases. Let

$$K(\delta) = \int_0^\delta H(u)\,du$$

The following assumption is needed concerning it.

ASSUMPTION 5.3. Ξ is a permissible subset of $L_1(Q_\rho)$ with $K(1)$ finite.

Here "permissible" is the notion that Pollard (1984) introduced to step across measurability issues.

This form of assumption is also made by LeCam (1986).

Let $U_\rho(\Xi)$ denote the set of real, bounded functions on Ξ that are uniformly continuous with respect to the metric ρ. This will be the space in which the sample paths of X^n and its limit lie.

The following definition may be found in Pollard (1989), Section 9, for example.

DEFINITION 5.1. The sequence of processes $\{X^n(a),\, a \in \Xi\}$ is said to converge in distribution to the process $\{X(a),\, a \in \Xi\}$ if

$$E^*\{g(X^n)\} \to E\{g(X)\}$$

for every $g \in U_\rho(\Xi)$. (Here E^* stands for the outer expectation.)

Now the principal theorem of this section may be stated.

THEOREM 5.1. *Under Assumptions* 5.1, 5.2 *and* 5.3, *the process X^n converges in distribution to a process X. The limit process is concentrated on $U_\rho(\Xi)$ and is Gaussian with mean* 0 *and covariance kernel* (5.1).

Under further regularity conditions one can bound the difference $E\{\tilde{J}^n(a) - \tilde{J}(a)\}$ and study the limit process centered at $\tilde{J}(a)$.

It may be worthwhile to conceptualize the result in other fashions. LeCam (1986), Section 16.7, in an empirical process context indicates the existence of a Gaussian process Z^n_* with the same mean and covariance as an X^n and an X^n_* with the same law as X^n, but on the same probability space as Z^n_* such that $\{\sup |X^n_*(a) - Z^n_*(a)| : a \in \Xi\} \to 0$ in probability. Such a result appears the more useful in thinking about finite n approximations. See also LeCam (1989).

6. DISCUSSION AND EXTENSIONS

It would seem that Theorem 5.1 may be extended to the nonGaussian case in the manner of Dahlhaus (1988), with a change in the covering numbers requirement. This paper has been concerned with the case of a real-valued process. Extensions to the complex-valued and r-vector-valued cases are immediate. As concern is with the details of a particular case, results have not been phrased in terms of time parameter in for example a local field, although some such extensions are directly available. Following Chapter 3 in Gelfand *et al.* (1969) there may be interesting extensions to the groups of adeles and ideles. In a later paper extensions to p-adic valued processes will be set down.

PROOFS

Proof of Lemma 4.1. One sees that the cumulant in question is given by

$$\int_{|t_j|_p \leqslant p^n} \cdots \int \exp\left(-2\pi i\langle\lambda_1 t_1 + \ldots + \lambda_k t_k\rangle\right) c_k(t_1 - t_k, \ldots, t_{k-1} - t_k)\, dt_1 \ldots dt_k$$

The result follows on making the change of variables $u_j = t_j - t_k$, $t = t_k$ For $j = 1, \ldots, k-1$ and noting that the region $\{|t|_p, |u_j + t|_p \leqslant p^n\}$ is equivalent to the region $\{|t|_p, |u_j|_p \leqslant p^n\}$ following the (unusual) properties of the p-adic norm.

Proof of Theorem 4.1. See the proof of Theorem 4.2.

Proof of Theorem 4.2. One simply notes, following expression (4.2), that the joint cumulant of order k is $o(p^n)$ and so when the standardization $p^{-n/2}$ is introduced tends to 0 for $k = 3, 4, \ldots$.

Proof of Theorem 4.3. Arguing exactly as in the proof of Theorem 2 of Brillinger (1968) one has that, under the indicated conditions,

$$\operatorname{cum}\{\tilde{J}^n(a_1), \ldots, \tilde{J}^n(a_k)\}$$

is $o(p^{-n(k-1)})$. In consequence the standardized cumulants of order greater than 2 tend to 0 and the asymptotic normality follows.

Next we note the following theorem of Pollard (1989), Section 10.

THEOREM A.1. *Let* $\{X^n(a) : a \in \Xi\}$ *be stochastic processes indexed by a totally bounded pseudometric space* (Ξ, ρ). *Suppose*: (i) *the finite dimensional distributions of the variates* $\{X^n(a_1), \ldots, X^n(a_k)\}$ *converge in distribution for each* k; (ii) *for each* $\epsilon > 0$ *and* $\eta > 0$ *there is a* $\delta > 0$ *such that*

$$\limsup P^*\{\sup_{\rho(a-b)<\delta} |X^n(a) - X^n(b)| > \eta\} < \epsilon \qquad \text{(A.1)}$$

Then there exists a Borel measure P *concentrated on* $U_\rho(\Xi)$, *with finite dimensional distributions given by the limits of* (i), *such that* X^n *tends to* P *in distribution.*

The difficult step in the proof of Theorem 5.1 will be to verify the equicontinuity condition (A.1). The next theorem sets an approach up.

The following theorem is proved just as Theorem 1 in Bentkus *et al.*

(1975), with a slight twist to be mentioned after the statement. (See also Statulevicius (1977).)

THEOREM A.2. *Under Assumption 5.2, re the process $Y(.)$, and assuming that $a(.)$ is absolutely integrable, for any $y > 0$*

$$\text{Prob}\,\{|\tilde{J}^n(a) - E\{\tilde{J}^n(a)\}| \geqslant y\} \leqslant \exp\,\{-D_1 y^2 p^n/[4(D_2 + y)]\} \quad \text{(A.2)}$$

where $D_1 = 1/[\rho(a)\rho(c_2)]$ and $D_2 = \rho(a)c_2(0)$.

The twist is that in Bentkus *et al.* (1975), in expression (19) one bounds out the term "$k = 1$" rather than the term "$j = 1$". From the theorem follows

COROLLARY. *Under the assumptions of the theorem, for any $\eta > 0$,*

$$\text{Prob}\,\{|X^n(a)| \geqslant \eta\rho(a)\} \leqslant 2\exp\,\{-\eta^2/[4\rho(c_2)(c_2(0) + \eta p^{-n/2})]\} \quad \text{(A.3)}$$

for $n = 1, 2$.

Proof of Theorem 5.1. Theorem A.1 will be employed. That the finite dimensional distributions converge to the appropriate Gaussian distributions follows from Theorem 4.3. It remains to demonstrate (A.1). We will proceed more or less in the fashion of Dahlhaus (1988), who more or less follows Pollard (1984), Section VII.2.

Write $c(t) = a(t) - b(t)$, then $X^n(a) - X^n(b) = X^n(c)$. From the Corollary to Theorem A.2

$$\text{Prob}\,\{|X^n(c)| > \eta\rho(c)\} \leqslant \Psi(\eta)$$

for n sufficiently large, where $\Psi(.)$ is defined by (5.2). Next, arguing as in Lemma VII.2 of Pollard (1984), under the stated conditions and noting that the assumed boundedness of $Y(.)$ on U_n means that $X^n(.)$ has continuous sample paths for n sufficiently large,

$$\text{Prob}\,\{|X^n(c)| > 26K(\rho(c)) \text{ for some } a, b \text{ in } \Xi \text{ with } \rho(c) \leqslant \epsilon\} \leqslant 2\epsilon \quad \text{(A.4)}$$

for all $0 < \epsilon < 1$.

Now

$$\text{Prob}\,\{\sup_{\rho(c)<\delta} |X^n(c)| > \eta\}$$

is bounded by the sum of

$$\text{Prob}\,\{|X^n(c)| > \eta \text{ and } \eta > 26K(\rho) \text{ for some } a, b \text{ with } \rho(c) \leqslant \delta\}$$

and an expression of the same form, but with the ">26" replaced by "$\geqslant 26$". Following (A.4) the first probability is bounded by 2δ, which is $\leqslant \epsilon$ for $\delta \leqslant \epsilon/2$ and the second is bounded by Prob $\{\eta \leqslant 26K(\delta)\}$. Noting Assumption 5.3 this last will be 0 for sufficiently small δ.

Noting the dependency of the right-hand side of (A.3) on n, another possible route to the equicontinuity condition would be via the "restricted chaining" approach of Pollard (1984).

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