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# Asymptotic Theory of Estimates of k-Th Order Spectra

## I. Introduction and Summary of Results

There is a growing literature on the k-th order spectra of stationary vector-valued processes and questions relating to their estimation (see [1-2],[4],[6],[8], [9], [11-12].) This paper is concerned with the asymptotic theory of estimates of k-th order spectra. A class of estimates is considered that is of an elementary form computationally and yet leads to asymptotic results of a simple form. The asymptotic mean and variance of these estimates is investigated as well as the covariances and joint asymptotic normality of several estimates of the same or differing orders. A basic property the processes are assumed to possess is a tendency for values of the process, well separated in time, to be approximately statistically independent. There is also an accompanying paper on applications. (See "Computation and Interpretation of kth order spectra" by D. R. Brillinger and M. Rosenblatt in this volume).

## II. Background

### A. Stationary Processes

We will be concerned with stationary, r-vector valued (column vector) processes $X(t) = (X_a(t); a = 1,\dots,r)$ with real-valued components. It will be convenient to assume that all moments exist. (We do not necessarily assume $EX(t) = 0$.) Time $t$ will be assumed to run through the integers.

Stationarity with respect to second order moments implies that the process $X(t)$ has a vector-valued Fourier representation

(2.1) $$X(t) = \int_{-\pi}^{\pi} \exp\{it\lambda\}\, dZ(\lambda),$$

in mean square with $Z(\lambda) = (Z_a(\lambda); a = 1,\dots,r)$ a process with





orthogonal increments, that is for $-\pi < \lambda,\ \mu < \pi$,

(2.2) $$E dZ(\lambda) dZ(\mu)' = \delta(\lambda + \mu)\, dF(\lambda)$$

where $A'$ denotes the transpose of $A$ with $F$ an $r \times r$ non-decreasing, bounded, matrix-valued, Hermitian function, and $\delta(\lambda)$ is the Dirac delta function. Thus the matrix-valued covariance function of

(2.3) $$EX(t_1)X(t_2)' = \int_{-\pi}^{\pi} \exp\{i(t_1 - t_2)\lambda\}\, dF(\lambda).$$

The fact that $X(t)$ is assumed to have real-valued components implies that

(2.4) $$\overline{dZ(-\lambda)} = dZ(\lambda)$$

and therefore that

(2.5) $$\overline{dF(-\lambda)} = dF(\lambda).$$

Since $t$ takes on integer values, $Z_a(\lambda)$ may be defined for values of $\lambda$ outside the interval $(-\pi, \pi)$ by the relation

(2.6) $$Z_a(\lambda) = Z_a(\lambda + 2j\pi)$$

for $j = 0, \pm 1, \pm 2, \dots$.
(These results are due to Bochner and Cramer and may be found in [3].) Let us define

(2.7) $$\eta(\lambda) = \sum_{-\infty}^{\infty} \delta(\lambda + 2j\pi).$$

The existence of all moments and the full stationarity imply that the moments satisfy

(2.8) $$m_{a_1,\dots,a_k}(t_1,\dots,t_k) = EX_{a_1}(t_1)\dots X_{a_k}(t_k) = m_{a_1,\dots,a_k}(t+t_1,\dots,t+t_k)$$

for $t = 0, \pm 1, \pm 2, \dots$. It is convenient to assume (see [1]) that the moments $m$ have Fourier representations

$$(2.9)\quad m_{a_1,\ldots,a_k}(t_1,\ldots,t_k) = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\{i\sum_{1}^{k} t_j w_j\} E\{\prod_1^k dZ_{a_j}(w_j)\},$$

where

$$(2.10)\quad E\{\prod_1^k dZ_{a_j}(w_j)\} = \eta(w_1+\ldots+w_k)\, dG_{a_1,\ldots,a_k}(w_1,\ldots,w_k)$$

and $G$ is of bounded variation with $dG$ zero unless $\Sigma_1^k w_j = 0, \pm 2\pi, \ldots$. Such a representation without $G$ of bounded variation is most likely true if the transform is interpreted in an appropriate sense (see [4]). With $G$ of bounded variation, it is not true in all generality, but is valid for a wide and interesting class of stationary processes. Let $c_{a_1,\ldots,a_k}(t_1,\ldots,t_k)$ denote the joint cumulant of $X_{a_1}(t_1),\ldots, X_{a_k}(t_k)$. Then the assumption of (2.10) is equivalent to

$$(2.11)\quad c_{a_1\ldots a_k}(t_1,\ldots,t_k) = c_{a_1\ldots a_k}(t+t_1,\ldots,t+t_k)$$
$$= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\{i\sum_1^k t_j w_j\} c\{dZ_{a_j}(w_j);\quad j=1,\ldots,k\}$$

with the cumulant

$$(2.12)\quad c\{dZ_{a_j}(w_j);\ j=1,\ldots,k\} = \eta(w_1+\ldots+w_k)\, dF_{a_1\ldots a_k}(w_1,\ldots,w_k),$$

where $F$ is of bounded variation with $dF$ zero unless $\Sigma_1^k w_j = 0, \pm 2\pi, \ldots$. It will be convenient to assume that the cumulants $c_{a_1\ldots a_k}$ are in $L_1$ as a function of some $(k-1)$-tuple of t's (and hence any other $(k-1)$-tuple.) This implies that $F$ is differentiable on the manifold $\Sigma_1^k w_j = 0, \pm 2\pi, \ldots$ with

$$(2.13)\quad dF_{a_1\ldots a_k}(w_1,\ldots,w_k)\,\eta(w_1+\ldots+w_k)$$
$$= f_{a_1\ldots a_k}(w_1,\ldots,w_k)\,\eta(w_1+\ldots+w_k)\, dw_1\ldots dw_k.$$

It is convenient to write $f$ as a function of $k$ variables even though it is zero off $\Sigma_1^k w_j = 0, \pm 2\pi, \ldots$. Notice that $f$ is continuous on



$\Sigma_1^k w_j = 0, \pm 2\pi, \ldots$.

The stationarity of the process, primarily a time domain concept, corresponds precisely to the condition that in the Fourier analysis of k-th order moments (or cumulants) the spectral mass is located on the principal manifold $\Sigma_1^k w_j \equiv 0 \pmod{2\pi}$ of k-dimensional wave number space for every integer $k > 0$. Further, one can readily see that

$$(2.14)\quad dG_{a_1\ldots a_k}(w_1,\ldots,w_k)\,\eta(w_1+\ldots+w_k)$$
$$= f_{a_1\ldots a_k}(w_1,\ldots,w_k)\,\eta(w_1+\ldots+w_k)\, dw_1\ldots dw_k$$

as long as the wave number vector $(w_1,\ldots,w_k)$ does not lie in a proper submanifold of $\Sigma_1^k w_j \equiv 0 \pmod{2\pi}$ of the form

$$(2.15)\quad \sum_{j\in J} w_j \equiv 0 \pmod{2\pi}$$

where $J$ is a nonvacuous proper subset of $1,\ldots,k$. The discussion of what happens on (or near) such proper submanifolds of the principal manifold in spectral analysis has to contain some detail and is somehwat more complicated than the usual situation on the principal manifold, but off proper submanifolds. The fact that the process has real-valued components implies that

$$(2.16)\quad f_{a_1\ldots a_k}(w_1,\ldots,w_k) = \overline{f}_{a_1\ldots a_k}(-w_1,\ldots,-w_k).$$

There are also symmetries introduced if some components are repeated in the computation of higher order moments or spectra. A detailed discussion of these symmetries and questions of aliasing for higher order spectra is given in the accompanying paper on applications.

B. Second Order Case.

At this point we shall briefly discuss the asymptotic properties of a class of estimates of second order spectra. The hope is that this will motivate and provide a base for the treatment of estimates of k-th order spectra. Assume that we observe a zero mean, discrete time parameter, real-valued stationary $X(t)$, $t = 0, \pm 1, \ldots$, from time $t = 0$ to time $t = T-1$. Certain features of the discussion will indicate what happens in both the discrete and continuous case, while others will be typical only of the discrete case; however, these will be pointed out at the appropriate place. Obvious estimates of the covariances $m(\tau) = EX(t)X(t+\tau)$ are given by

$$m^{(T)}(\tau) = T^{-1} \sum_{0 \le t,\, t+\tau \le T-1} X(t)X(t+\tau) \,. \tag{2.17}$$

The Fourier transform of this sequence is

$$I^{(T)}(\lambda) = (2\pi)^{-1} \sum_{\tau=-(T-1)}^{T-1} m^{(T)}(\tau) \exp\{-i\tau\lambda\} \tag{2.18}$$

$$= (2\pi T)^{-1} \Big| \sum_{t=0}^{T-1} X(t) \exp\{-it\lambda\} \Big|^2 ,$$

the periodogram. Asymptotic properties of the periodogram, under an additional condition like $\Sigma_{-\infty}^{\infty} |\tau|\,|m(\tau)| < \infty$, include

$$EI^{(T)}(\lambda) = f(\lambda) + 0(T^{-1}) \tag{2.19}$$

$$\operatorname{cov}\{I^{(T)}(\lambda), I^{(T)}(\mu)\} = \frac{f^2(\lambda)}{T^2} \left\{ \frac{\sin^2 \frac{T}{2}(\lambda+\mu)}{\sin^2 \frac{1}{2}(\lambda+\mu)} + \frac{\sin^2 \frac{T}{2}(\lambda-\mu)}{\sin^2 \frac{1}{2}(\lambda-\mu)} \right\} + 0(T^{-1}) . \tag{2.20}$$

Clearly the periodogram is not a consistent estimate (in the sense of mean square convergence) of $f(\lambda)$, even though it is asymptotically unbiased. However, the asymptotic orthogonality of $I^{(T)}(\lambda)$, $I^{(T)}(\mu)$ for $\lambda \ne \mu$ $0 \le \lambda, \mu < \pi$ suggests that one would obtain a reasonable estimate by smoothing. We shall smooth by using a sequence of weight functions $W_T(u)$ derived from a fixed weight function $W(u)$. Let $W(u)$ be a given weight function that is bounded, non-negative, symmetric about zero ($W(u) = W(-u)$) and such that $W(u) \to 0$ as $|u| \to \infty$. Further let $\int W(u)\,du = 1$. Let

$$W_T(u) = K_T B_T^{-1} W(B_T^{-1} u) \quad \text{if } |u| \le \pi \tag{2.21}$$

with $K_T$ a suitable renormalization constant so that $\int_{-\pi}^{\pi} W_T(u)\,du = 1$.

$B_T$ is chosen so that $B_T \to 0$ as $T \to \infty$, but $TB_T \to \infty$ as $T \to \infty$. Notice that this implies that $K_T \to 1$ as $T \to \infty$. For $u$ outside of $(-\pi, \pi)$, $W_T(u)$ is to be taken as a periodic function with period $2\pi$. An estimate $f^{(T)}(\lambda)$ of $f(\lambda)$ is then given by



$$f^{(T)}(\lambda) = \int_{-\pi}^{\pi} W_T(\lambda-\alpha) I^{(T)}(\alpha)\,d\alpha \,. \tag{2.22}$$

The alternative way of writing $f^{(T)}(\lambda)$ as

$$f^{(T)}(\lambda) = (2\pi)^{-1} \sum_{\tau=-(T-1)}^{T-1} w_T(\tau) m^{(T)}(\tau) \exp\{-i\tau\lambda\} \tag{2.23}$$

where

$$w_T(\tau) = \int_{-\pi}^{\pi} W_T(u) \exp\{i\tau u\}\,du \tag{2.24}$$

may often be more convenient for computation. It is natural to call $B_T$ the bandwidth of the function $W_T(u)$. The estimate is asymptotically unbiased as is the periodogram

$$Ef^{(T)}(\lambda) = f(\lambda) + 0(T^{-1}) \,. \tag{2.25}$$

A discussion of the bias $b_T(\lambda) = Ef^{(T)}(\lambda) - f(\lambda)$ can be found in [7]. The estimate is asymptotically consistent since

$$\operatorname{cov}\{f^{(T)}(\lambda), f^{(T)}(\mu)\} \tag{2.26}$$

$$= 2\pi T^{-1}\Big\{ \int_{-\pi}^{\pi} W_T(\lambda-\alpha) W_T(\mu+\alpha) f^2(\alpha)\,d\alpha + \int_{-\pi}^{\pi} W_T(\lambda-\alpha) W_T(\mu-\alpha) f^2(\alpha)\,d\alpha \Big\} + 0(T^{-1}) .$$

For fixed $\lambda, \mu$ with $\lambda \ne \mu$ $(0 \le \lambda, \mu < \pi)$

$$\operatorname{cov}\{f^{(T)}(\lambda), f^{(T)}(\mu)\} = 0(T^{-1}) \text{ as } T \to \infty .$$

Thus $f^{(T)}(\lambda)$, $f^{(T)}(\mu)$ are asymptotically uncorrelated if $\lambda \ne \mu$, $0 \le \lambda, \mu < \pi$. Now consider what this tells us about the asymptotic behavior of the variance

$$\operatorname{var} f^{(T)}(\lambda) = 2\pi T^{-1}\Big\{ \int_{-\pi}^{\pi} W_T(\lambda-\alpha) W_T(\lambda+\alpha) f^2(\alpha)\,d\alpha + \int_{-\pi}^{\pi} W_T^2(\lambda-\alpha) f^2(\alpha)\,d\alpha \Big\} + 0(T^{-1}) . \tag{2.27}$$

Clearly,

$$\lim_{T\to\infty} B_T T \operatorname{var} f^{(T)}(\lambda) = 2\pi f^2(\lambda)\eta_{2\lambda}\int W^2(u)\,du \tag{2.28}$$

if $0 \le \lambda \le \pi$, where if $\delta_\lambda$ denotes the Kronecker delta,

$$\eta_\lambda = \sum_{-\infty}^{\infty} \delta_{\lambda+2j\pi} \,. \tag{2.29}$$

Notice that in expression (2.28) one has a doubling of the variance at $\lambda = 0$ and $\lambda = \pi$. The doubling of the variance at $\lambda = 0$ happens in both the discrete and continuous case, while the doubling of the variance at $\lambda = \pi$ is characteristic of the discrete case only. If we were to write the spectral density in the form given in (16), we would have

$$f(\lambda) = f(w_1, w_2) \tag{2.30}$$

with $w_1 = \lambda = -w_2$ and then it is seen that $\lambda = 0$ corresponds to the submanifold $(0,0)$ of $w_1 + w_2 = 0$. Formula (2.28) is unpleasant in that the asymptotic behavior at $\lambda = 0, \pi$ appear as discontinuities. Formula (2.27) is much more informative in this respect since it indicates that the transition between the usual asymptotic behavior and that at $\lambda = 0, \pi$ takes place in intervals about $\lambda = 0, \pi$ whose length is of the order of magnitude of the bandwidth $B_T$. The estimates $f^{(T)}(\lambda_j)$ of $f(\lambda_j)$, $0 \le \lambda_j \le \pi$, $j = 1, \ldots, s$ are asymptotically normally distributed with means and covariances given by (2.25) and (2.26) to the first order as $T \to \infty$, under appropriate conditions.

## C. The k-th Order Case

Because of the stationarity the moments $m_{a_1 \ldots a_k}(t_1, \ldots, t_k)$ and cumulants $c_{a_1 \ldots a_k}(t_1, \ldots, t_k)$ depend only on $k-1$ appropriately chosen variables. At times, such a representation will be convenient though perhaps unesthetic. The representation will depend on the index of the time variable one uses as a base point in forming time differences. If $t_j$ is used as the base point we shall write

$$\begin{aligned} {}_j m_{a_1 \ldots a_k}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k) \\ = m_{a_1 \ldots a_k}(t+v_1, \ldots, t+v_{j-1}, t, t+v_{j+1}, \ldots, t+v_k) \end{aligned} \tag{2.31}$$

with a corresponding definition for ${}_j c_{a_1 \ldots a_k}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_k)$.



Notice that the $k$ functions of $k-1$ variables formed in this way from the initially given function of $k$ variables will generally look quite different. There are however, certain obvious relations between them enabling one to determine any one in terms of any other. These relations are commented on in greater detail in the accompanying paper on computation and interpretation. Whenever an assumption in some result is stated in terms of such a representation, the assumption will have the same form in terms of any such representation. For convenience we will write $m'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})$ for ${}_k m_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})$ with a similar definition for $c'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})$. In the frequency domain, the contracted form

$$\begin{aligned} {}_j f_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_k) \\ = f_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \lambda_{j+1}, \ldots, \lambda_k) \end{aligned} \tag{2.32}$$

where $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$, will at times be used. We shall also write $f'_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1})$ for ${}_k f_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1})$ on occasion.

In deriving results we will make a basic assumption concerning the nature of the process $X(t)$.

Assumption I. Given the strictly stationary process $X(t) = (X_a(t);\ a = 1, \ldots, r)$ we assume

$$\sum_{v_1, \ldots, v_{k-1} = -\infty}^{\infty} |v_j c'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})| < \infty \tag{2.33}$$

for $j = 1, \ldots, k-1$ and any k-tuple $a_1, \ldots, a_k$ when $k = 2, 3, \ldots$ .

This assumption relates directly to the smoothness of the k-th order spectra. In fact we can prove that

$$\begin{aligned} |f_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j + \mu_j, \lambda_{j+1}, \ldots, \lambda_k) \\ - f_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{j-1}, \lambda_j, \lambda_{j+1}, \ldots, \lambda_k)| \\ \le |\mu_j| (2\pi)^{-k+1} \sum_v |v_j c'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})| \end{aligned} \tag{2.34}$$

or alternatively we may note that (2.33) implies that $f'_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1})$ has a uniformly continuous, uniformly bounded gradient.

Before going on to state some results of interest, we shall have to introduce some additional notation. Let

$$d_a^{(T)}(\lambda) = \sum_{t=0}^{T-1} X_a(t) \exp\{-i\lambda t\}. \tag{2.35}$$

Moments of products of such sums as

$$\prod_{j=1}^{k} d_{a_j}^{(T)}(\lambda_j) \tag{2.36}$$

will be of special interest to us. Notice that the familiar second order periodogram is simply,

$$I^{(T)}(\lambda) = (2\pi T)^{-1} |d^{(T)}(\lambda)|^2. \tag{2.37}$$

We shall have occasion to introduce higher order analogues of the second order periodogram. In the following lemma the joint cumulant of $d_{a_1}^{(T)}(\lambda_1), \ldots, d_{a_k}^{(T)}(\lambda_k)$ is estimated. Let

$$\Delta_T(\lambda) = \sum_{t=0}^{T-1} \exp\{-i\lambda t\} \tag{2.38}$$

$$= \exp\{i\lambda(T-1)/2\} \sin \lambda T/2 \big/ \sin \lambda/2.$$

We note that $\Delta_T(\lambda) = T$ if $\lambda \equiv 0 \pmod{2\pi}$ and $\Delta_T(\lambda) = 0$ if $\lambda = 2\pi n/T$, $n$ an integer not equal to $0, \pm T, \ldots$.

Lemma 1.* Suppose that

$$\sum_{v_1, \ldots, v_{k-1} = -\infty}^{\infty} |v_j c'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})| < \infty \tag{2.39}$$

for $j = 1, \ldots, k-1$; then the cumulant

* The proofs of Lemmas and Theorems may be found in Section IV.



$$c\{d_{a_1}^{(T)}(\lambda_1), \ldots, d_{a_k}^{(T)}(\lambda_k)\} = (2\pi)^{k-1} \Delta_T\left(\sum_1^k \lambda_j\right) f'_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1}) + O(1) \tag{2.40}$$

where the error term $O(1)$ is uniform for all $\lambda_1, \ldots, \lambda_k$.

We note that the cumulant reduces to $O(1)$ if for $j = 1, \ldots, k$, $\lambda_j = 2\pi n_j/T$, $n_j$ an integer, but $\Sigma_1^k \lambda_j \not\equiv 0 \pmod{2\pi}$. This reduction will greatly assist us in deriving statistical properties of the proposed estimates.

The expression

$$I_{a_1 \ldots a_k}^{(T)}(\lambda_1, \ldots, \lambda_k) = (2\pi)^{-k+1} T^{-1} \prod_{j=1}^{k} d_{a_j}^{(T)}(\lambda_j) \tag{2.41}$$

where $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$ is a k-th order analogue of the second order periodogram. This is suggested by the fact that

$$I_{a_1 \ldots a_k}^{(T)}(\lambda_1, \ldots, \lambda_k) \tag{2.42}$$

$$= (2\pi)^{-k+1} \sum_{v_1 = -T+1}^{T-1} \cdots \sum_{v_{k-1} = -T+1}^{T-1} m_{a_1 \ldots a_k}^{(T)}(v_1, \ldots, v_{k-1}) \exp\left\{-i \sum_1^{k-1} v_j \lambda_j\right\}$$

with

$$m_{a_1 \ldots a_k}^{(T)}(v_1, \ldots, v_{k-1}) = T^{-1} \sum_{\substack{0 \le t \le T-1 \\ 0 \le t+v_1 \le T-1 \\ \vdots \\ 0 \le t+v_{k-1} \le T-1}} X_{a_1}(t+v_1) \ldots X_{a_{k-1}}(t+v_{k-1}) X_{a_k}(t), \tag{2.43}$$

since $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$. We shall call $I_{a_1 \ldots a_k}^{(T)}(\lambda_1, \ldots, \lambda_k)$ a k-th order periodogram and it is always to be understood that the sum of its $k$ variables satisfies $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$.

In connection with the k-th order periodogram we can prove

Lemma 2. Let $X(t) = (X_a(t); a = 1, \ldots, r)$ be a strictly stationary process satisfying Assumption I. The k-th order periodogram, $I_{a_1 \ldots a_k}^{(T)}(\lambda_1, \ldots, \lambda_k)$, given by (2.41) is such that

$$(2.44)\qquad E I^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) = f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) + O(T^{-1})$$

provided that the $\lambda_1,\ldots,\lambda_k$ do not lie in any proper submanifold of the principal manifold. (The term $O(T^{-1})$ is not uniform here.) The expected value typically diverges as $T\to\infty$ if the $\lambda$'s do lie in a proper submanifold. If the process has zero mean,

$$(2.45)\qquad T^{-k+2}\{E I^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) I^{(T)}_{a_{k+1}\ldots a_{2k}}(\lambda_{k+1},\ldots,\lambda_{2k})$$

$$-E I^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) E I^{(T)}_{a_{k+1}\ldots a_{2k}}(\lambda_{k+1},\ldots,\lambda_{2k})\}$$

$$= \sum \frac{\sin\frac{1}{2}T(\lambda_{i_1}+\lambda_{i_2})}{T\sin\frac{1}{2}(\lambda_{i_1}+\lambda_{i_2})}\cdots\frac{\sin\frac{1}{2}T(\lambda_{i_{2k-1}}+\lambda_{i_{2k}})}{T\sin\frac{1}{2}(\lambda_{i_{2k-1}}+\lambda_{i_{2k}})}$$

$$\cdot f'_{a_{i_1}a_{i_2}}(\lambda_{i_1})\cdots f'_{a_{i_{2k-1}}a_{i_{2k}}}(\lambda_{i_{2k-1}}) + O(T^{-1})$$

where $\Sigma$ extends over all groupings $\{(i_1,i_2),\ldots,(i_{2k-1},i_{2k})\}$ of $(1,\ldots,2k)$ into pairs such that for some $j$ one of $i_{2j-1}, i_{2j}$ is in the range $1$ to $k$ and one in the range $k+1$ to $2k$.

From (2.44) we see that the k-th order periodogram is asymptotically unbiased off submanifolds; however from (2.45) we see that it is not in general consistent. From (2.45) we also see that if $\lambda_{i_1}+\lambda_{i_2},\ldots,\lambda_{i_{2k-1}}+\lambda_{i_{2k}}$ are not all congruent to $0(\bmod 2\pi)$, then the estimates are asymptotically orthogonal. This suggests that we may obtain a reasonable estimate by performing a smoothing avoiding submanifolds. We have already noted that (2.33) implies the continuity of $f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ as a function of the $\lambda$'s on $\Sigma_1^k\lambda_j \equiv 0(\bmod 2\pi)$. Consequently we would expect that taking an average of values near, but not too near, a submanifold may provide a reasonable estimate on the submanifold. Both off and on submanifolds we have been led to estimates that are weighted averages of the k-th order periodogram avoiding the submanifolds.

We can now describe a class of estimates of k-th order spectra that are analogous to the estimates of second order spectra discussed in the previous section. Let $W(u_1,\ldots,u_k)$ be a given



continuous weight function on the plane $\Sigma_1^k u_j = 0$ and such that

$$(2.46)\qquad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W(u_1,\ldots,u_k)\,\delta\Big(\sum_1^k u_j\Big)\,du_1\ldots du_k = 1$$

and

$$(2.47)\qquad W(-u_1,\ldots,-u_k) = W(u_1,\ldots,u_k)\ .$$

Let

$$(2.48)\qquad W_T(u_1,\ldots,u_k) = B_T^{-k+1} W(B_T^{-1}u_1,\ldots,B_T^{-1}u_k)$$

for all $u_j$ with $\Sigma_1^k u_j = 0$. If the resulting series is uniformly and absolutely convergent, set

$$(2.49)\qquad \overline{W}_T(u_1,\ldots,u_k) = \sum W_T(u_1+2j_1\pi,\ldots,u_k+2j_k\pi)$$

for $\Sigma_1^k u_j \equiv 0(\bmod 2\pi)$ with $|u_j| \le \pi$, $j=1,\ldots,k$ and where the summation in (2.49) extends over $j_1,\ldots,j_k$ with $u_1+\ldots+u_k+2\pi(j_1+\ldots+j_k) = 0$. $B_T$ is chosen so that $B_T\to 0$ as $T\to\infty$, but $B_T^{k-1}T\to\infty$.

An estimate of $f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ that one could consider off the submanifolds is given by

$$(2.50)\qquad f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_T(\lambda_1-\alpha_1,\ldots,\lambda_k-\alpha_k) I^{(T)}_{a_1\ldots a_k}(\alpha_1,\ldots,\alpha_k)\,\delta\Big(\sum_1^k\alpha_j\Big)\,d\alpha_1\ldots d\alpha_k$$

$$= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \overline{W}_T(\lambda_1-\alpha_1,\ldots,\lambda_k-\alpha_k) I^{(T)}_{a_1\ldots a_k}(\alpha_1,\ldots,\alpha_k)\,\eta\Big(\sum_1^k\alpha_j\Big)d\alpha_j\ \ d\alpha_1\ldots d\alpha_k,$$

where as usual $\Sigma_1^k\lambda_j \equiv 0(\bmod 2\pi)$. This can also be written in the form

(2.51) $$f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) = (2\pi)^{-k+1}\sum_{v_1=-T+1}^{T-1}\cdots\sum_{v_{k-1}=-T+1}^{T-1} w_T(v_1,\ldots,v_{k-1}) \cdot m^{(T)}_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})\exp\{-i\sum_1^{k-1} v_j\lambda_j\}$$

where the $w_T(v_1,\ldots,v_{k-1})$ are the Fourier coefficients of $W_T(u_1,\ldots,u_{k-1})$,

(2.52) $$w_T(v_1,\ldots,v_{k-1}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_T(u_1,\ldots,u_k) \cdot \exp\{i\sum_1^{k-1} v_j u_j\}\delta(\sum_1^k u_j)\, du_1\ldots du_k .$$

The asymptotic behavior of estimates of this form, $k = 3$, and when the series have zero mean have been discussed in some detail in [9] and [13].

For estimates on the submanifolds, the earlier discussion indicated that we could average estimates of the form of (2.50) for $\lambda$'s in a neighborhood of a submanifold, but not actually on it. An alternative is to employ an expression of the form

(2.53) $$(2\pi)^{-k+1}\sum_{v_1=-T+1}^{T-1}\cdots\sum_{v_{k-1}=-T+1}^{T-1} w_T(v_1,\ldots,v_{k-1}) \cdot c^{(T)}_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})\exp\{-i\sum_1^{k-1} v_j\lambda_j\}$$

where $c^{(T)}_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})$ is an estimate of the cumulant $c'_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})$.

We could base an estimate of $c'_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})$ upon estimates of moments of the form of expression (2.43). Alternatively we could base moment estimates upon the associated circular process, that is use moment estimates of the form

(2.54) $$\hat{m}^{(T)}_{a_1\ldots a_k}(v_1,\ldots,v_{k-1}) = T^{-1}\sum_{t=0}^{T-1}\hat{X}_{a_1}(t+v_1)\ldots\hat{X}_{a_{k-1}}(t+v_{k-1})\hat{X}_{a_k}(t)$$



where $|v_j| \le T-1$ and $\hat{X}(s) = X(s)$ if $0 \le s \le T-1$, while it is the periodic extension, with period $T$, outside this range. Estimates of this second form lead to estimates of k-th order spectra that are simpler both computationally and analytically. This simplicity is due, in part, to the fact that one is now in effect carrying out a discrete harmonic analysis on the discrete circle and more generally the discrete k-torus. The results obtained on the asymptotic behavior of these estimates constitute a natural extension of those obtained in the papers cited above for $k = 3$, with the added advantage that a simpler variance formula is obtained. We consider a sequence of weight functions $w_T(u_1,\ldots,u_k)$ just as those in formula (2.48). However, the estimate is obtained by summing discretely. We define $\Phi(u_1,\ldots,u_k)$ to equal 1 if $\Sigma_1^k u_j \equiv 0 (\mathrm{mod}\ 2\pi)$, but $\sum_{j\in J} u_j \not\equiv 0(\mathrm{mod}\ 2\pi)$ where $J$ is any nonvacuous proper subset of $1,\ldots,k$ and $\Phi(u_1,\ldots,u_k) = 0$ otherwise. The proposed estimate is now given by

(2.55) $$f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) = (2\pi)^{k-1}T^{-k+1}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty} W_T(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}) \cdot \Phi(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}) I^{(T)}_{a_1\ldots a_k}(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T})$$
$$= (2\pi)^{k-1}T^{-k+1}\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1}\overline{W}_T(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}) \cdot \Phi(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}) I^{(T)}_{a_1\ldots a_k}(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T})$$

where $\Sigma_1^k\lambda_j \equiv 0(\mathrm{mod}\ 2\pi)$. Notice that estimates of the type (2.55) differ from these of the type (2.50) in that any contribution from a proper submanifold is supressed by the function $\Phi$. Later, in proving one of the main asymptotic results, we will see that this modification leads to an asymptotic variance simpler than one might expect. If we wish we may normalize the weights in (2.55) by dividing through by

(2.56) $$K_T = \sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty} W_T(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T})\Phi(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T})$$

or some similar expression. This will not affect the asymptotic results of the paper, since under the conditions to be postulated for $W$, $K_T = 1 + O(B_T^{-1}T^{-1})$. Finally, we will see that the estimate $f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ given by (2.55) may also be written

$$(2.58)\quad f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) = (2\pi)^{k-1}T^{-k+1}$$

$$\cdot \sum_{v_1=-T+1}^{T-1} \cdots \sum_{v_{k-1}=-T+1}^{T-1} \hat{w}_T(v_1,\ldots,v_{k-1})\hat{c}'_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})\exp\{-i\sum_1^{k-1} v_j\lambda_j\}$$

where $\hat{c}'_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})$ is an estimate of $c'_{a_1\ldots a_k}(v_1,\ldots,v_{k-1})$ derived from the moment estimates of expression (2.54) and

$$(2.59)\quad \hat{w}_T(v_1,\ldots,v_{k-1})$$

$$= (2\pi)^{k-1}T^{-k+1} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} W_T(\lambda_1 - \frac{2\pi s_1}{T},\ldots,\lambda_k - \frac{2\pi s_k}{T})$$

$$\cdot \exp\{i \sum_1^{k-1} v_j(\lambda_j - \frac{2\pi s_j}{T})\}\eta_{\sum_1^k s_j} = w_T(v_1,\ldots,v_{k-1}) + O(B_T^{-1}T^{-1}).$$

## III. Statement of Some Typical Results

### A. The asymptotic mean of a k-th order spectral estimate

The class of estimates we will be concerned with are those given by expression (2.55) with $W_T$ given by expression (2.48). With respect to the function $W$ we will assume

Assumption II. There exist $A$, $\epsilon > 0$ such that

$$(3.1)\quad |W(u_1,\ldots,u_{k-1}, -\sum_1^{k-1} u_j)| \le A(1+[\sum_1^{k-1} u_j^2]^{\frac{1}{2}})^{-(k+\epsilon-1)}$$

and



$$(3.2)\quad |\frac{\partial}{\partial u_\ell} W(u_1,\ldots,u_{k-1}, -\sum_1^{k-1} u_j)| \le A(1 + [\sum_1^{k-1} u_j^2]^{\frac{1}{2}})^{-(k+\epsilon-1)}$$

for $\ell = 1,\ldots,k-1$.

(3.2) will be needed when we come to approximate certain finite sums by integrals.

We can prove

Theorem 1. Let $X(t) = (X_a(t);\ a = 1,\ldots,r)$ be a strictly stationary process satisfying Assumption I. Let $f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ be an estimate of $f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ of the type given in expression (2.55) with the weight function $W$ satisfying Assumption II. If $B_T T \to \infty$ with $B_T \to 0$ as $T \to \infty$, then

$$(3.3)\quad Ef^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_T(\lambda_1-\alpha_1,\ldots,\lambda_k-\alpha_k) f_{a_1\ldots a_k}(\alpha_1,\ldots,\alpha_k)\eta(\sum_1^k a_j)d\alpha_1\ldots d\alpha_k + O(B_T^{-1}T^{-1}).$$

We see that up to $O(B_T^{-1}T^{-1})$ the expected value of the proposed estimate is a weighted average of the k-th order spectrum with weight concentrated in a neighborhood of the wave numbers of interest. The theorem has the following immediate corollary

Corollary. Under the conditions of the theorem

$$(3.4)\quad \lim_{T\to\infty} Ef^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k) = f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k),$$

that is, the estimate is asymptotically unbiased.

By strengthening the conditions on $W$ and $f_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k)$ we can obtain an alternative expression for the expected value given in (3.3). This alternative expression is illuminating in several ways. Suppose $W$ is such that

$$(3.5)\quad \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} |u_\ell|^P |W(u_1,\ldots,u_k)|\delta(\sum_1^k u_j)du_1\ldots du_k < \infty$$

for some integer $P \ge 1$ and $\ell = 1,\ldots,k-1$.

Let

(3.6) $$w(v_1,\dots,v_{k-1}) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W(u_1,\dots,u_{k-1}) \exp\{i\sum_1^{k-1} v_j u_j\}\,\delta(\sum_1^k u_j)\,du_1\dots du_k .$$

We can prove

Theorem 2. Let $X(t) = (X_a(t);\ a = 1,\dots,r)$ be a strictly stationary process such that

$$\frac{\partial^p}{\partial\lambda_1^{p_1}\cdots\partial\lambda_{k-1}^{p_{k-1}}} f'_{a_1\dots a_k}(\lambda_1,\dots,\lambda_{k-1})$$

is uniformly bounded for $\Sigma_1^{k-1} p_j = p$, $p_j \geq 0$, $p \leq P$. Let $f^{(T)}_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k)$ be an estimate of $f_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k)$ of the type given in expression (2.55) with the weight function $W$ satisfying Assumption II. If $B_T T \to \infty$ as $B_T \to 0$ and $T \to \infty$, then

(3.7) $$E f^{(T)}_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k)$$
$$= \sum_{p=0}^{P-1} \sum_{p_1,\dots,p_{k-1}} \frac{B_T^p}{i^p p!} \frac{\partial^p}{\partial\lambda_1^{p_1}\cdots\partial\lambda_{k-1}^{p_{k-1}}} f'_{a_1\dots a_k}(\lambda_1,\dots,\lambda_{k-1})$$
$$\cdot \frac{\partial^p}{\partial\lambda_1^{p_1}\cdots\partial\lambda_{k-1}^{p_{k-1}}} w(0,\dots,0) + O(B_T^P) + O(B_T^{-1}T^{-1}) .$$

We note that we may eliminate bias up to order $B_T^{P-1}$ by selecting a weight function such that

(3.9) $$\frac{\partial^p}{\partial\lambda_1^{p_1}\cdots\partial\lambda_{k-1}^{p_{k-1}}} w(0,\dots,0) = 0$$



for $p = 1,\dots,P-1$. We note that the term $O(B_T^P) + O(B_T^{-1}T^{-1})$ collapses to $O(B_T^P)$ if $B_T^{P+1}T \to \infty$ as $T \to \infty$. Finally we note that the partial derivatives mentioned in the theorem will be uniformly bounded if

(3.10) $$\sum_{v_1=-\infty}^{\infty}\cdots\sum_{v_{k-1}=-\infty}^{\infty} |v_j|^P |c'_{a_1\dots a_k}(v_1,\dots,v_{k-1})| < \infty$$

for $j = 1,\dots,k-1$.

B. Asymptotic covariance of k-th order spectral estimates.

Continuing to examine asymptotic properties of estimates of the form (2.55) we can prove

Theorem 3. Let $X(t) = (X_a(t);\ a = 1,\dots,r)$ be a strictly stationary process satisfying Assumption I. Let $f^{(T)}_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k)$ and $f^{(T)}_{a'_1\dots a'_k}(\mu_1,\dots,\mu_k)$ be estimates of the type given in formula (2.55) of $f_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k)$ and $f_{a'_1\dots a'_k}(\mu_1,\dots,\mu_k)$ respectively with weight function $W$ satisfying Assumption II. If $B_T^{k-1}T \to \infty$ as $B_T \to 0$ and $T \to \infty$, then

(3.11) $$\operatorname{cov}[f^{(T)}_{a_1\dots a_k}(\lambda_1,\dots,\lambda_k), f^{(T)}_{a'_1\dots a'_k}(\mu_1,\dots,\mu_k)]$$
$$= 2\pi T^{-1} \sum_{\underline{P}} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_T(\lambda_1-\beta_1,\dots,\lambda_k-\beta_k)$$
$$\cdot W_T(\mu_1+\beta_1,\dots,\mu_k+\beta_k) \prod_1^k \eta(a_j+\beta_{\underline{P}(j)})\,\eta(\sum_1^k \alpha_j) \prod_1^k f_{a_j, a'_{\underline{P}(j)}}(\alpha_j)$$
$$\cdot d\alpha_1\dots d\alpha_k\, d\beta_1\dots d\beta_k + O(B_T^{-k+2}T) ,$$

where the summation is over all permutations $\underline{P}$ on the integers $1,\dots,k$ and the error term is uniform in the $\lambda$'s and $\mu$'s subject to

$$\sum_1^k \lambda_j \equiv 0 \ (\mathrm{mod}\ 2\pi) \quad \text{and} \quad \sum_1^k \mu_j \equiv 0 \ (\mathrm{mod}\ 2\pi) .$$

Notice that the random variables in formula (3.11) are generally complex-valued. The covariance of two complex-valued random variables, X, Y is taken to be

$$(3.12) \qquad \mathrm{cov}(X, Y) = EXY^* - EXEY^* .$$

In the present situation it is enough to derive these complex covariances because the covariances of the real and imaginary parts of the estimates may then be derived in an elementary manner. The asymptotic results on covariances can be given in the following less informative but simpler form.

Corollary. Under the assumptions of Theorem 3,

$$(3.13) \quad \lim_{T\to\infty} B_T^{k-1} T \,\mathrm{cov}[f^{(T)}_{a_1\ldots a_k}(\lambda_1,\ldots,\lambda_k), f^{(T)}_{a_1'\ldots a_k'}(\mu_1,\ldots,\mu_k)]$$

$$= 2\pi \sum_{\underline{P}} \eta_{\lambda_1-\mu_{\underline{P}(1)}} \cdots \eta_{\lambda_k-\mu_{\underline{P}(k)}} f_{a_1 a'_{\underline{P}(1)}}(\lambda_1)\ldots f_{a_k a'_{\underline{P}(k)}}(\lambda_k)$$

$$\cdot \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} W(\tau_1,\ldots,\tau_k) W(\tau_{\underline{P}(1)},\ldots,\tau_{\underline{P}(k)})\, \delta(\sum_1^k \tau_j)\, d\beta_1\ldots d\beta_k ,$$

where the summation is over all permutations $P$ on the integers $1,\ldots,k$. In particular if one orders the series so that the wave numbers satisfy $-\pi < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \pi$ and $-\pi < \mu_1 < \mu_2 < \ldots < \mu_k < \pi$ and one is off proper submanifolds of the principal manifold the expression on the right of formula (3.13) becomes

$$(3.14) \quad 2\pi \prod_{j=1}^{k} \{\eta_{\lambda_j-\mu_j} f'_{a_j a'_j}(\lambda_j)\} \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} W^2(\beta_1,\ldots,\beta_k)\, \delta(\sum_1^k \beta_j)\, d\beta_1\ldots d\beta_k .$$

Alternatively if $W$ is symmetric in its $k$ arguments and we are dealing with a single series (3.13) becomes

(3.15)

$$2\pi f(\lambda_1)\ldots f(\lambda_k) \sum_P \eta_{\lambda_1-\mu_{\underline{P}(1)}} \cdots \eta_{\lambda_k-\mu_{\underline{P}(k)}} \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} W^2(\beta_1,\ldots,\beta_k)\, \delta(\sum_1^k \beta_j)\, d\beta_1\ldots d\beta_k .$$

In the case of (3.14) we note that we may standardize the estimate by dividing it by $(\Pi_1^k f_{a_j}(\lambda_j))^{1/2}$. Also we note that (3.15) contains fewer terms in the case $k = 3$ than formula (7.10) of [9]. This simplification is caused by the subtraction of estimates of



lower order product moments. One can see that generally in the case where these lower order moments are known exactly, as in [9], one obtains a simpler formula by persisting in subtracting their sample estimates.

In both the theorem and the corollary the expressions for the principal contribution to the covariance involve only the second order spectra. Intuitively, this should not be completely surprising, since in forming estimates of k-th order spectra one is still effectively narrow band filtering, though in a more complicated situation, and narrow band filtered series tend to become Gaussian. This point is discussed in "Computation and Interprelation of kth order spectra" by D. R. Brillinger and M. Rosenblatt in this volume.

The corollary above is the analogue of the result given in formula (2.28) for estimates of second order spectra. The apparent discontinuity in formula (3.13) for the limiting covariance on proper submanifolds of the principal manifold parallels the discontinuities at zero and $\pi$ in formula (2.28). It is for this reason that the uniform result given in Theorem 3 is more satisfactory. Formula (3.11) indicates that there is for finite, but large, $T$ a continuous transition between the typical result off a submanifold and the typical result on a submanifold in a region of linear bandwidth $B_T$ about the submanifold.

In the next section, to complete the covariance analysis, we will see that estimates of a k-th order and an $\ell$-th order, $k \neq \ell$, spectrum are asymptotically uncorrelated under suitable regularity conditions.

## C. Asymptotic distribution of the estimates.

Suppose we now have estimates of cumulant spectra of orders $k_1 \leq k_2 \leq \ldots \leq k_b$ of the form given in (2.55) with scale factors $B_T^{(1)} \leq \ldots \leq B_T^{(b)}$. Write the j-th such estimate in the form

$$(3.16) \quad f^{(T)}_{A_j}(\lambda^{(j)}) = \left(\frac{2\pi}{T}\right)^{k_j-1} \sum W_T^{(j)}\left(\lambda^{(j)} - \frac{2\pi s^{(j)}}{T}\right) \Phi\left(\frac{2\pi s^{(j)}}{T}\right) I^{(T)}_{A_j}\left(\frac{2\pi s^{(j)}}{T}\right)$$

where $A_j$ designates the indices of the $k_j$ series involved in the j-th estimate. We can prove

Theorem 4. Let $X(t) = (X_a(t); a = 1,\ldots,r)$ be a strictly stationary process satisfying Assumption I. Let $f^{(T)}_{A_j}(\lambda^{(j)})$, $j = 1,\ldots,b$, be estimates as given by formula (3.16) of orders $k_1 \leq \ldots \leq k_b$ whose weight functions $W^{(j)}$ satisfy Assumption II. The bandwidths $B_T^{(j)}$ of the estimates are assumed to satisfy

$$B_T^{(j)} \to 0, \quad B_T^{(j)k_j - 1} T \to \infty$$

as $T \to \infty$ with $B_T^{(1)} \leq \ldots \leq B_T^{(b)}$. Bandwidths of estimates of the same order are taken to be equal. Under these assumptions the estimates are asymptotically jointly normally distributed as $T \to \infty$ with estimates of different orders asymptotically independent and estimates of the same order having covariance structure given by (3.13).

D. Extensions. The results stated above are given for discrete parameter processes. One can obtain parallel results for continuous time parameter, continuous in mean square, processes. Assumption I is replaced by

Assumption I'. Given the strictly stationary process $X(t) = (X_a(t); a = 1, \ldots, r)$ we assume

(3.17) $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |v_j c'_{a_1 \ldots a_k}(v_1, \ldots, v_{k-1})| \, dv_1 \ldots dv_{k-1} < \infty$$

for $j = 1, \ldots, k-1$, and any k-tuple $a_1, \ldots, a_k$ when $k = 2, 3, \ldots$.

Expression (2.35) is replaced by

(3.18) $$d_a^{(T)}(\lambda) = \int_0^T X_a(t) \exp\{-i\lambda t\} \, dt$$

and expression (2.38) by

(3.19) $$\Delta_T(\lambda) = \int_0^T \exp\{-i\lambda t\} dt .$$

The definition of the k-th order periodogram, and the estimate of the k-th order cumulant spectrum remain the same. Turning to Theorems 1 through 4, the results remain valid with the simple replacement of $\eta$ by $\delta$ throughout as the proofs given extend directly. One may also extend the results to processes with a multidimensional time parameter.

## IV. Proofs

### A. Cumulants.

Throughout the proofs of the theorems of this paper, extensive use will be made of the joint cumulants of polynomial functions



of random variables. As a result we will need an algorithm that may be employed in the derivation of such cumulants. The algorithm we will describe, and the notation we will use, is that of [5].

Consider a (not necessarily rectangular) two way table

(4.1) $$\begin{matrix} (1,1) & \ldots & (1,k_1) \\ \vdots & \ldots & \vdots \\ (J,1) & \ldots & (J,k_J) \end{matrix}$$

and a partition of its elements into disjoint sets, $\{P_1, P_2, \ldots, P_m\}$. We shall say that two sets of the partition, $P_{i_1}$ and $P_{i_2}$, hook if there exist $(j_1, j_2) \in P_{i_1}$ and $(j_3, j_4) \in P_{i_2}$ such that $j_1 = j_3$. We shall say that the sets $P_{i'}$ and $P_{i''}$ communicate if there exists a sequence of sets $P_{i_1} = P_{i'}, P_{i_2}, \ldots, P_{i_r} = P_{i''}$ such that $P_{i_j}$ and $P_{i_{j+1}}$ hook for each $j$. A partition is said to be indecomposable if all its sets communicate. One can see that if the rows of table (4.1) are denoted by $R_1, \ldots, R_J$ then $\{P_1, \ldots, P_m\}$ is indecomposable if and only if there exist no sets $P_{i_1}, \ldots, P_{i_n}$, $(n < m)$ and rows $R_{j_1}, \ldots, R_{j_p}$, $(p < J)$ with

(4.2) $$P_{i_1} \cup \ldots \cup P_{i_n} = R_{j_1} \cup \ldots \cup R_{j_p} .$$

Turning to the calculation of cumulants one may prove

Lemma I [5]. Given an array $\|y_{mn}\|$, $n = 1, \ldots, k_m$, $m = 1, \ldots, J$ of random variables, consider the $J$ complex-valued random variables

(4.3) $$z_m = \prod_{n=1}^{k_m} y_{mn} .$$

The joint J-th order cumulant $c\{z_1, \ldots, z_J\}$ is given by

$$\sum_{\nu} C_{\nu_1} \ldots C_{\nu_p}$$

where $C_\nu = c\{y_{a_1}, \ldots, y_{a_m}\}$ when $\nu = (a_1, \ldots, a_m)$, the a's being pairs of integers taken from the table (4.1), and the summation in (4.4) extends over all indecomposable partitions of (4.1).

The proof given in [5] is for real random variables, however the extension to complex random variables is immediate.

As an application of this lemma we see that

$$E(y_1,\dots,y_k) = \sum_{\nu} C_{\nu_1}\cdots C_{\nu_p} \tag{4.5}$$

where the summation extends over all partitions $(\nu_1,\dots,\nu_p)$ of the integers $(1,\dots,k)$ and $C_\nu$ denotes the joint cumulant of the $y$'s with subscripts in $\nu$. The relation (4.5) may be inverted to obtain

$$c\{y_1,\dots,y_k\} = \sum (-1)^{p-1}(p-1)!\,\mu_{\nu_1}\cdots\mu_{\nu_p} \tag{4.6}$$

where the summation again extends over all partitions $(\nu_1,\dots,\nu_p)$ of the integers $(1,\dots,k)$, but now $\mu_\nu$ denotes the product moment of the $y$'s with subscripts in $\nu$.

The final algebraic property of cumulants that we will require is

$$c\{\sum_{n_1} a_{1n_1} y_{1n},\dots,\sum_{n_J} a_{Jn_J} y_{Jn}\} = \sum_{n_1}\cdots\sum_{n_J} a_{1n_1}\cdots a_{Jn_J}\, c\{y_{1n},\dots,y_{Jn}\} \tag{4.7}$$

where the $a_{ij}$ are constants.

Proof of Lemma 1. Set $\lambda = \lambda_1+\dots+\lambda_k$, $t_\alpha = -\min(u_1,\dots,u_{k-1},0)$ and $t_\beta = T-1-\max(u_1,\dots,u_{k-1},0)$ if for $|u_j|\le T-1$, $j=1,\dots,k-1$ we have $0\le t_\alpha \le t_\beta \le T-1$. Set ranges of summations involving these limits to the empty set if this is not the case.

$$c\{d_{a_1}^{(T)}(\lambda_1),\dots,d_{a_k}^{(T)}(\lambda_k)\} \tag{4.8}$$

$$= \sum_{t_1=0}^{T-1}\cdots\sum_{t_k=0}^{T-1} \exp\{-i\sum_1^k \lambda_j t_j\}\, c'_{a_1\dots a_k}(t_1,\dots,t_k)$$

$$= \sum_{u_1=-T+1}^{T-1}\cdots\sum_{u_{k-1}=-T+1}^{T-1} \exp\{-i\sum_1^{k-1}\lambda_j u_j\}\, c'_{a_1\dots a_k}(u_1,\dots,u_{k-1}) \sum_{t=t_\alpha}^{t_\beta} \exp\{-i\lambda t\}.$$

We note that $\Sigma_{t=0}^{T-1} \exp\{-i\lambda t\} = \Delta_T(\lambda)$,

$$|\sum_{t=0}^{t_\alpha -1} \exp\{-i\lambda t\}| \le |t_\alpha| \le |u_1| + \dots + |u_{k-1}|,$$

and

$$|\sum_{t=t_\beta+1}^{T-1} \exp\{-i\lambda t\}| \le |u_1| + \dots + |u_{k-1}|. \tag{4.9}$$

We see that

$$|f'_{a_1\dots a_k}(\lambda_1,\dots,\lambda_{k-1}) - (2\pi)^{-k+1} \tag{4.10}$$

$$\cdot \sum_{u_1=-T+1}^{T-1}\cdots\sum_{u_{k-1}=-T+1}^{T-1} \exp\{-i\sum_1^{k-1}\lambda_j u_j\}\, c'_{a_1\dots a_k}(u_1,\dots,u_{k-1})|$$

$$\le (2\pi)^{-k+1}\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} T^{-1}(|u_1|+\dots+|u_{k-1}|)\,|c'_{a_1\dots a_k}(u_1,\dots,u_{k-1})| = O(T^{-1}).$$

The stated result follows on writing (4.9) in the form

(4.11)

$$\sum_{-T+1}^{T-1}\cdots\sum_{-T+1}^{T-1} \exp\{-i\sum_1^{k-1}\lambda_j u_j\}\, c'_{a_1\dots a_k}(u_1,\dots,u_{k-1})[(\sum_0^{T-1} - \sum_0^{t_\alpha -1} - \sum_{t_\beta+1}^{T-1} \exp\{-i\lambda t\}].$$

Proof of Lemma 2.

We are interested in

$$E\,I_{a_1\dots a_k}^{(T)}(\lambda_1,\dots,\lambda_k) = (2\pi)^{-k+1}T^{-1}E\prod_{j=1}^{k} d_{a_j}^{(T)}(\lambda_j) \tag{4.12}$$

where $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$. Taking $y_j = d_{a_j}^{(T)}(\lambda_j)$ we may use (4.5) to obtain an expression for the expected value at issue in terms of the joint cumulants of the $d_{a_j}^{(T)}(\lambda_j)$ and the reader will remember that we obtained an expression for these joint cumulants in Lemma I. In (4.13) below we will sum across all partitions $(\nu_1,\dots,\nu_p)$ of $(1,\dots,k)$, $a_\nu = \{a_j | j\in\nu\}$ $\tilde\lambda_\nu = \Sigma_{j\in\nu}\lambda_j$, and $\lambda'_\nu = \{\lambda_j | j\in\nu$, but the last $\lambda$ is supressed$\}$. We see that

$$E\,I_{a_1\dots a_k}^{(T)}(\lambda_1,\dots,\lambda_k) \tag{4.13}$$

$$= (2\pi)^{-k+1}T^{-1}\sum_\nu (2\pi)^{k-p}[\Delta_T(\tilde\lambda_{\nu_1}) f'_{a_{\nu_1}}(\lambda'_{\nu_1}) + O(1)]\dots[\Delta_T(\tilde\lambda_{\nu_p}) f'_{a_{\nu_p}}(\lambda'_{\nu_p}) + O(1)].$$

In (4.13), if $\nu = (1, \ldots, k)$ for example,

$$(4.14) \qquad \Delta_T(\tilde{\lambda}_\nu) f'_{a_\nu}(\lambda'_\nu) = \Delta_T(\lambda_1 + \ldots + \lambda_k) f'_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1}) .$$

We now see that (2.44) follows from (4.13) since the $\lambda$'s lie in no submanifold.

Turning to the proof of (2.45), from Lemma I, given random variables $y_1, \ldots, y_{2k}$,

$$(4.15) \qquad Ey_1 \ldots y_{2k} - Ey_1 \ldots y_k \, Ey_{k+1} \ldots y_{2k} = \sum_\nu C_{\nu_1} \ldots C_{\nu_p}$$

where the summation extends over the indecomposable partitions of the table

$$(4.16) \qquad \begin{matrix} 1 & \ldots & k \\ k+1 & \ldots & 2k \end{matrix}$$

and $C_\nu$ denotes the joint cumulant of the $y$'s with subscripts in $\nu$. Setting $y_j = d^{(T)}_{a_j}(\lambda_j)$, $j = 1, \ldots, 2k$, the expression at issue in (2.45) may be written

$$(4.17) \qquad T^{-k+2}(2\pi)^{-2k+2} T^{-2} \sum_\nu (2\pi)^{k-p} [\Delta_T(\tilde{\lambda}_{\nu_1}) f'_{a_{\nu_1}}(\lambda'_{\nu_1}) + O(1)] \ldots [\Delta_T(\tilde{\lambda}_{\nu_p}) f'_{a_{\nu_p}}(\lambda'_{\nu_p}) + O(1)]$$

where the sum is over indecomposable partitions $\{\nu_1, \ldots, \nu_p\}$ selected from the table (4.16). The typical term in (4.17) has the form

$$(4.18) \quad T^{-k} \Delta_T(\tilde{\lambda}_{\nu_1}) \ldots \Delta_T(\tilde{\lambda}_{\nu_m}) f'_{a_{\nu_1}}(\lambda'_{\nu_1}) \ldots f'_{a_{\nu_m}}(\lambda'_{\nu_m}) \, O(1) = O(T^{m-k}) .$$

Noting that we have assumed (for the present only) that the series has mean zero, we see that the principal term occurs for $m = p = k$. The stated result now follows directly.

In the proofs of the theorems we will require a lemma allowing us to approximate discrete sums by integrals.

<u>Lemma 3. Consider a function</u> $g(u_1, \ldots, u_k)$ <u>with the property that there exist</u> $A, \epsilon > 0$ <u>such that</u>



$$(4.19) \quad |g(u_1, \ldots, u_k)|, \left|\frac{\partial g(u_1, \ldots, u_k)}{\partial u_\ell}\right| \le A\left(1 + \left[\sum_1^k u_j^2\right]^{\frac{1}{2}}\right)^{-(k+\epsilon)} = M(u_1, \ldots, u_k) .$$

<u>Given finite</u> $A_j \le B_j$, $j = 1, \ldots, k$ <u>and</u> $h_j > 0$, <u>let</u> $N_j = (B_j - A_j)/h_j$ <u>be an integer, then</u>

$$(4.20) \qquad h_1 \ldots h_k \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_k=0}^{N_k-1} g(A_1 + h_1 n_1, \ldots, A_k + h_k n) = \int_{A_1}^{B_1} \ldots \int_{A_k}^{B_k} g(u_1, \ldots, u_k)\, du_1 \ldots du_k + R$$

<u>where</u> $|R| \le (\Sigma_1^k h_j) K$ <u>with</u> $K$ <u>bounded and depending only on</u> $k$.

<u>Proof of Lemma</u> 3.

$$(4.21) \quad \left(\prod_1^k h_j\right) g(A_1 + h_1 n_1, \ldots, A_k + h_k n_k) - \int_{A_j + h_j n_j}^{A_j + h_j(n_j+1)} \ldots \int g(u_1, \ldots, u_k)\, du_1 \ldots du_k$$
$$= \int_{A_j + h_j n_j}^{A_j + h_j(n_j+1)} \ldots \int [g(A_1 + h_1 n_1, \ldots, A_k + h_k n_k) - g(u_1, \ldots, u_k)]\, du_1 \ldots du_k$$

and so

(4.22)

$$\left| \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_k=0}^{N_k-1} \left(\prod_1^k h_j\right) g(A_1 + h_1 n_1, \ldots, A_k + h_k n_k) - \int_{A_1}^{B_1} \ldots \int_{A_k}^{B_k} g(u_1, \ldots, u_k)\, du_1 \ldots du_k \right|$$
$$\le \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_k=0}^{N_k-1} \int_{A_j + h_j n_j}^{A_j + h_j(n_j+1)} \ldots \int |g(A_1 + h_1 n_1, \ldots, A_k + h_k n_k) - g(u_1, \ldots, u_k)|\, du_1 \ldots du_k$$
$$\le \sum_{n_1^t=0}^{N_1-1} \ldots \sum_{n_k^t=0}^{N_k-1} \int_{A_j + h_j n_j}^{A_j + h_j(n_j+1)} \ldots \int \left(\sum_1^k h_j\right) M(A_1 + h_1 n_1^t, \ldots, A_k + h_k n_k^t)\, du_1 \ldots du_k$$

where $n_j^t = n_j$ if $A_j + h_j n_j$ is nearer the origin than $A_j + h_j(n_j+1)$ and $n_j^t = n_j+1$ otherwise. We see that (4.22) does not exceed

$$(4.23)\quad \sum_{n_1^t=0}^{N_1-1} \cdots \sum_{n_k^t=0}^{N_k-1} M(A_1+h_1 n_1^t, \ldots, A_k+h_k n_k^t)\left(\sum_1^k h_j\right) \prod_1^k h_j$$

$$\leq 2^k \left(\sum_1^k h_j\right) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} M(u_1, \ldots, u_k)\, du_1 \ldots du_k$$

giving the stated result.

<u>Proof of Theorem 1.</u>

We are interested in the expected value of expression (2.55). This is given by

$$(4.24)\quad (2\pi)^{k-1} T^{-k+1} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} W_T\left(\lambda_1 - \frac{2\pi s_1}{T}, \ldots, \lambda_k - \frac{2\pi s_k}{T}\right)$$

$$\cdot\, \Phi\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right) E\, I^{(T)}_{a_1 \ldots a_k}\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right)$$

$$= (2\pi)^{k-1} T^{-k+1} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} W_T\left(\lambda_1 - \frac{2\pi s_1}{T}, \ldots, \lambda_k - \frac{2\pi s_k}{T}\right)$$

$$\cdot\, \Phi\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right) f_{a_1 \ldots a_k}\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right)$$

$$+ T^{-k} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} W_T\left(\lambda_1 - \frac{2\pi s_1}{T}, \ldots, \lambda_k - \frac{2\pi s_k}{T}\right) \Phi\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right) O(1)$$

from (4.13). The presence of $\Phi$ causes the disappearance of the $\Delta_T$'s after the first, because $\Phi = 0$ on any of the submanifolds. From Lemma 3, we see that



$$(4.25)$$

$$(2\pi)^{k-1} T^{-k+1} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} W_T\left(\lambda_1 - \frac{2\pi s_1}{T}, \ldots, \lambda_k - \frac{2\pi s_k}{T}\right) \Phi\left(\frac{2\pi s_1}{T}, \ldots, \frac{2\pi s_k}{T}\right)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_T(\lambda_1 - \alpha_1, \ldots, \lambda_k - \alpha_k)\, \eta\left(\sum_1^k \alpha_j\right) d\alpha_1 \ldots d\alpha_k + R$$

where $|R| \leq K T^{-1} B_T^{-1}$.

We saw that $f_{a_1 \ldots a_k}$ had uniformly bounded first derivatives; consequently the product $W f_{a_1 \ldots a_k}$ continues to satisfy the boundedness conditions of Lemma 3 and we see that the first term on the right hand side is given by

$$(4.26)$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_T(\lambda_1 - \alpha_1, \ldots, \lambda_k - \alpha_k) f_{a_1 \ldots a_k}(\alpha_1, \ldots, \alpha_k)\, \eta\left(\sum_1^k \alpha_j\right) d\alpha_1 \ldots d\alpha_k + O(B_T^{-1} T^{-1}).$$

The proof of the theorem is now completed. Since

$$(4.27)$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_T(\lambda_1 - \alpha_1, \ldots, \lambda_k - \alpha_k) f_{a_1 \ldots a_k}(\alpha_1, \ldots, \alpha_k)\, \eta\left(\sum_1^k \alpha_j\right) d\alpha_1 \ldots d\alpha_k$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W(\alpha_1, \ldots, \alpha_k) f_{a_1 \ldots a_k}(\lambda_1 - B_T \alpha_1, \ldots, \lambda_k - B_T \alpha_k)\, \eta\left(\sum_1^k \alpha_j\right) d\alpha_1 \ldots d\alpha_k.$$

The corollary follows from a standard convergence theorem.

<u>Proof of Theorem 2.</u>

Under the conditions of the theorem,

$$(4.28)\quad f'_{a_1 \ldots a_k}(\lambda_1 - B_T \alpha_1, \ldots, \lambda_{k-1} - B_T \alpha_{k-1})$$

$$= \sum_{p=0}^{P-1} \sum_{p_1, \ldots, p_{k-1}} \frac{(-B_T)^p}{p!} \frac{\partial^p}{\partial \lambda_1^{p_1} \ldots \partial \lambda_{k-1}^{p_{k-1}}} f'_{a_1 \ldots a_k}(\lambda_1, \ldots, \lambda_{k-1})\, \alpha_1^{p_1} \ldots \alpha_{k-1}^{p_{k-1}}$$

$$+ B_T^P \left(\sum_1^{k-1} |\alpha_j|^P\right) R$$

where R is bounded. The stated result follows on substituting from (4.28) into (4.27) and noting that

$$(4.29)\qquad \frac{1}{i^p}\frac{\partial^p}{\partial\lambda_1^{p_1}\cdots\partial\lambda_{k-1}^{p_{k-1}}}w(0,\dots,0)$$

$$=(-1)^p\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\alpha_1^{p_1}\cdots\alpha_{k-1}^{p_{k-1}}W(\alpha_1,\dots,\alpha_k)\,\delta\Big(\sum_1^k\alpha_j\Big)d\alpha_1\cdots d\alpha_k .$$

Proof of Theorem 3.

The covariance in question is given by

$$(4.30)\qquad (2\pi)^{2k-2}T^{-2k+2}\sum_{r_1=-\infty}^{\infty}\cdots\sum_{r_k=-\infty}^{\infty}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty}$$

$$\cdot\, W_T\Big(\lambda_1-\frac{2\pi s_1}{T},\dots,\lambda_k-\frac{2\pi s_k}{T}\Big)W_T\Big(-\mu_1-\frac{2\pi r_1}{T},\dots,-\mu_k-\frac{2\pi r_k}{T}\Big)$$

$$\cdot\,\Phi\Big(\frac{2\pi s_1}{T},\dots,\frac{2\pi s_k}{T}\Big)\Phi\Big(\frac{2\pi r_1}{T},\dots,\frac{2\pi r_k}{T}\Big)(2\pi)^{-2k+2}$$

$$\cdot\, T^{-2}\Big[E\prod_1^k d_{a_j}^{(T)}\Big(\frac{2\pi s_j}{T}\Big)\prod_1^k d_{a_j}^{(T)}\Big(\frac{2\pi r_j}{T}\Big)-E\prod_1^k d_{a_j}^{(T)}\Big(\frac{2\pi s_j}{T}\Big)E\prod_1^k d_{a_j'}^{(T)}\Big(\frac{2\pi r_j}{T}\Big)\Big] .$$

In the proof of Lemma 2 we saw in (4.15) that the expression in square brackets was given by $\Sigma_\nu C_{\nu_1}\cdots C_{\nu_p}$, where the summation extends over the indecomposable partitions of the table (4.16), $C_\nu$ is the joint cumulant of the $y_j$ with $j\in\nu$, and

$$y_j=d_{a_j}^{(T)}\Big(\frac{2\pi s_j}{T}\Big)\ \text{for}\ j=1,\dots,k\ \text{and}\ y_{j+k}=d_{a_j'}^{(T)}\Big(\frac{2\pi r_j}{T}\Big) .$$

The expression in square brackets may therefore be written

(4.31)

$$\sum_\nu(2\pi)^{2k-p}\Big[\Delta_T\Big(\frac{2\pi\tilde q_{\nu_1}}{T}\Big)f'_{b_{\nu_1}}\Big(\frac{2\pi q'_{\nu_1}}{T}\Big)+O(1)\Big]\cdots\Big[\Delta_T\Big(\frac{2\pi\tilde q_{\nu_p}}{T}\Big)f'_{b_{\nu_p}}\Big(\frac{2\pi q'_{\nu_p}}{T}\Big)+O(1)\Big]$$



where q denotes a collection of r's and s's and b a collection of a's and a''s. Due to the presence of the $\Phi$ terms in (4.30) any $\Delta_T$ term in (4.31) corresponding to a submanifold all in one row of (4.16) drops out. When (4.31) is substituted into (4.30), a characteristic term of the latter is

$$(4.32)\quad T^{-2k}\sum_{r_1=-\infty}^{\infty}\cdots\sum_{r_k=-\infty}^{\infty}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty}W_T\Big(\lambda_1-\frac{2\pi s_1}{T},\dots,\lambda_k-\frac{2\pi s_k}{T}\Big)$$

$$\cdot\, W_T\Big(-\mu_1-\frac{2\pi r_1}{T},\dots,-\mu_k-\frac{2\pi r_k}{T}\Big)\Phi\Big(\frac{2\pi s_1}{T},\dots,\frac{2\pi s_k}{T}\Big)\Phi\Big(\frac{2\pi r_1}{T},\dots,\frac{2\pi r_k}{T}\Big)(2\pi)^{2k-p}$$

$$\cdot\,\Delta_T\Big(\frac{2\pi\tilde q_{\nu_1}}{T}\Big)\cdots\Delta_T\Big(\frac{2\pi\tilde q_{\nu_\ell}}{T}\Big)f'_{b_{\nu_1}}\Big(\frac{2\pi q'_{\nu_1}}{T}\Big)\cdots f'_{b_{\nu_\ell}}\Big(\frac{2\pi q'_{\nu_\ell}}{T}\Big)+O(1) .$$

From Lemma 3 we see that this takes the form

$$(4.33)\qquad T^{-2k}(2\pi)^{-2k+\ell+1}T^{2k-\ell-1}(2\pi)^{2k-p}$$

$$\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}W_T(\lambda_1-\alpha_1,\dots,\lambda_k-\alpha_k)W_T(-\mu_1-\beta_1,\dots,-\mu_k-\beta_k)$$

$$\cdot\, T^\ell\eta(\tilde\gamma_{\nu_1})\cdots\eta(\tilde\gamma_{\nu_\ell})\eta\Big(\sum_1^k\alpha_j\Big)\eta\Big(\sum_1^k\beta_j\Big)f_{b'_{\nu_1}}(\gamma'_{\nu_1})\cdots f'_{b_{\nu_\ell}}(\gamma'_{\nu_\ell})d\alpha_1\cdots d\alpha_k d\beta_1\cdots d\beta_k$$

+ a lower order term,

where $\gamma$ denotes a collection of $\alpha$'s and $\beta$'s. We see that (4.33) is $O(B_T^{\ell+1}T)$. The principal term in (4.30) consequently occurs for $\ell=k$. Remembering that terms on submanifolds drop out, we see that the covariance is given by

$$(4.34)\quad 2\pi T^{-1} \sum_{\underline{P}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} W_T(\lambda_1-\alpha_1, \ldots, \lambda_k-\alpha_k)\, W_T(-\mu_1-\beta_1, \ldots, -\mu_k-\beta_k)$$

$$\cdot\, \eta(\alpha_1+\beta_{P(1)}) \cdots \eta(\alpha_{k-1}+\beta_{P(k-1)})\, \eta(\sum_1^k \alpha_j)\, \eta(\sum_1^k \beta_j)$$

$$\cdot \prod_1^k f'_{a_j a'_{\underline{P}(j)}}(\alpha_j)\, d\alpha_1 \ldots d\alpha_k\, d\beta_1 \ldots d\beta_k + O(B_T^{-k+2}T),$$

from which the stated result follows.

We turn to the proof of (3.13) of the corollary. Integrating out the $\beta$'s in (3.11) and then carrying out the substitution

$$(4.35)\quad \tau_j = B_T^{-1}(-\mu_j + \alpha_{\underline{P}^{-1}(j)} - 2\pi \ell_j),$$

we see that (3.11) may be written

$$(4.36)\quad \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_k=-\infty}^{\infty} 2\pi T^{-1} \sum_{\underline{P}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} B_T^{-k+1}$$

$$\cdot\, W(B_T^{-1}[\lambda_1-\mu_{\underline{P}(1)}+2\pi\ell_1] - \tau_{\underline{P}(1)}, \ldots, B_T^{-1}[\lambda_k-\mu_{\underline{P}(k)}+2\pi\ell_k] - \tau_{\underline{P}(k)})$$

$$\cdot\, W(\tau_1, \ldots, \tau_k)\, \eta(\sum_1^k \tau_j) \prod_1^k f'_{a_j a'_{\underline{P}(j)}}(\mu_{\underline{P}(j)} + B_T \tau_{\underline{P}(j)})\, d\tau_1 \ldots d\tau_k + O(B_T^{-k+2}T)$$

from which (3.13) follows under the stated regularity conditions.

Continuing with the proof of the corollary, (3.14) follows once one notes that under the conditions on the $\lambda$'s and $\mu$'s the summation over permutations in (3.13) contains but one term, the stated one. (3.15) now follows immediately under the stated conditions.

Proof of Theorem 4.

The theorem will be established if we can show that the correlation of estimates of different order tends to zero and that all cumulants of order three or more of the suitably normalized estimates tend to zero. It is clear from Theorem 3 that asymptotically the order of magnitude of the standard deviation of an estimate of order $k$ is $[B_T^{k-1}T]^{-\frac{1}{2}}$. We therefore normalize the j'th estimate by



dividing by the factor $[B_T^{(j)k_j-1}T]^{-\frac{1}{2}}$. The covariances and cumulants of these normalized estimates are computed.

Consider first the covariance of two normalized estimates of order $k_1 \le k_2$. The covariance can be written in a form paralleling (4.32)

$$(4.37)\quad T^{-k_1-k_2} T\, B_T^{(1)\frac{1}{2}(k_1-1)} B_T^{(2)\frac{1}{2}(k_2-1)}$$

$$\cdot \sum_{\underline{P}} \sum_{s^{(1)}, s^{(2)}} B_T^{(1)^{-k_1+1}} W^{(1)}(B_T^{(1)^{-1}}\{\lambda^{(1)} - \frac{2\pi s^{(1)}}{T}\}) B_T^{(2)^{-k_2+1}} W^{(2)}(B_T^{(2)^{-1}}\{\lambda^{(2)} - \frac{2\pi s^{(}}{T}$$

$$\cdot\, \Phi(\frac{2\pi s^{(1)}}{T})\, \Phi(\frac{2\pi s^{(2)}}{T})\, \Delta_T(\frac{2\pi \tilde{q}_{\nu_1}}{T}) \cdots \Delta_T(\frac{2\pi \tilde{q}_{\nu_\ell}}{T})\; O(1),$$

where one sums over all allowable partitions $\underline{P}$. The $\Delta_T$ terms provide $\ell$ linear restraints linking $s^{(2)}$ variables with $s^{(1)}$ variables. Keeping in mind the restraints on sum of $s^{(1)}$ variables, zero, and the sum of $s^{(2)}$ variables, zero, use these linear restraints to solve for an $s^{(2)}$ variable in terms of $s^{(1)}$ variables and possibly other $s^{(2)}$ variables. In this way either $\ell-1$ or $\ell$, $s^{(2)}$ variables are eliminated, depending on whether the $\ell$ linear restraints involve all $s^{(1)}$ and $s^{(2)}$ variables or only a proper subset. The product of the $\Delta_T$ terms contributes at most a factor $T^{\ell}$. Let $\delta$ be 1 or zero according as to whether the $\ell$ linear restraints involve all variables or not. By absorbing the factor $B_T^{(1)^{-k_1+1}} B_T^{(2)^{-k_2+1-\ell+\delta}} T^{-k_1-k_2+2+\ell+\delta}$ in the sum of the product of $W$ terms we can approximate it by a bounded integral. Here heavy use is made of the bounds in Assumptions I and II. We have yet to investigate the remaining factors in the sum corresponding to one such possible partition. They are

$$(4.38)\quad T^{-k_1-k_2+1}\, T^{k_1+k_2-2-\ell+\delta}\, T^{\ell} B_T^{(2)^{-\ell+\delta}} B_T^{(1)\frac{1}{2}(k_1-1)} B_T^{(2)\frac{1}{2}(k_2-1)}$$

$$= T^{-1+\delta} B_T^{(1)\frac{k_1-1}{2}} B_T^{(2)\frac{k_2-1}{2} - \ell + \delta}.$$

Now $n_2 \le k_1$ if $\delta = 1$ and $n_2 \le k_1 - 1$ if $\delta = 0$ since the partitions are irreducible. Since $B_T^{(1)} \le B_T^{(2)}$ with $B_T^{(j)} \to 0$ it is clear that if $\delta = 0$ every term goes to zero. If $\delta = 1$ all terms go to zero except for those with $n_2 = k_1 = k_2$.

Now let us consider the asymptotic behavior of cumulants of $J$ normalized estimates, $J \geq 3$. Such a cumulant can be written as

$$(4.39)\qquad T^{-k_1-\dots-k_J} T^{J/2} B_T^{(1)\frac{k_1-1}{2}} \dots B_T^{(J)\frac{k_J-1}{2}}$$

$$\cdot \sum_{\underline{P}} \sum_{s^{(i)}} B_T^{(1)^{-k_1+1}} W^{(1)}(B_T^{(1)^{-1}}\{\lambda^{(1)} - \frac{2\pi s^{(1)}}{T}\}) \dots B_T^{(J)^{-k_J+1}}$$

$$\cdot W^{(J)}(B_T^{(J)-1}\{\lambda^{(J)} - \frac{2\pi s^{(J)}}{T}\})\Phi(\frac{2\pi s^{(1)}}{T}) \dots \Phi(\frac{2\pi s^{(J)}}{T})\Delta_T(\frac{2\pi\tilde{q}_{\nu_1}}{T}) \dots \Delta_T(\frac{2\pi\tilde{q}_{\nu}}{T})O(1)$$

where one sums over all allowable partitions $\underline{P}$. We'd now like to show that for each partition $\underline{P}$ the sum over the $s^{(i)}$'s tends to zero. The $\Delta_T$ terms as before, provide $\ell$ linear restraints linking the $s^{(i)}$ variables, $i = 1, \dots, J$. Also the sum of the $s^{(i)}$ variables is zero for each $i = 1, \dots, J$. It will be convenient to refer to the rows of the table

$$(4.40)\qquad \begin{matrix} s_1^{(1)} \dots s_{k_1}^{(1)} \\ \dots \\ s_1^{(J)} \dots s_{k_J}^{(J)} \end{matrix} .$$

We say that a set of linear restraints ends on the $\alpha$th row if the variables with highest superscript involved are of the $\alpha$th row. The detailed argument that is given here is for the case of a partition in which all the elements in the $\alpha$th row are not in sets ending on the $\alpha$th row, $\alpha = 1, \dots, J-1$. We shall show that the sum over the $s^{(i)}$'s tends to zero for such partitions. For other partitions, similar but more elaborate arguments can be used. Let $\delta = 1$ or $0$ according as to whether all the $s$ variables or only a proper subset are involved in the $\ell$ linear restraints. Use the $\ell$ linear restraints to solve for variables of the highest superscript involved in terms of those of the same or lower superscript. In the $\alpha$th row there are initially $k_\alpha - 1$ free variables due to the condition on the sum of the $s^{(\alpha)}$ variables being zero, if $\alpha < J$. Let $n_\alpha$ be the number of linear restraints ending on the $\alpha$th row. Since all the variables on the $\alpha$th row are not governed by these restraints, it follows that we can eliminate $n_\alpha$ additional variables by the procedure described



above. Thus, there are $k_\alpha - 1 - n_\alpha$ free variables left in the $\alpha$th row, $\alpha < J$. In the Jth row a similar argument shows that there are $k_J - 1 + n_\alpha + \delta$ free variables left. If we account for a factor $T^{-1}B_T^{(\alpha)-1}$ for each free variable in the $\alpha$th row, the sum over the product of the W's aside from these factors can be approximated by a bounded integral using Assumption II on the W's and techniques used so often in previous theorems to replace a discrete sum by an integral. We have only to account for the remaining product of factors in terms of the $B_T$'s and T's. They amount to

$$(4.41)\qquad T^{-k_1-\dots-k_J} T^{J/2} B_T^{(1)\frac{k_1-1}{2}} B_T^{(2)\frac{k_2-1}{2}} \dots B_T^{(J)\frac{k_J-1}{2}}$$

$$\cdot T^{\ell}(TB_T^{(1)})^{k_1-1}(TB_T^{(2)})^{k_2-1-n_2} \dots (TB_T^{(J)})^{k_J-1-n_J+\delta} B_T^{(1)^{-k_1+1}} \dots B_T^{(J)^{-k_J+1}}$$

$$= T^{-J/2+\delta} B_T^{(1)\frac{k_1-1}{2}} B_T^{(2)\frac{k_2-1}{2}-n_2} \dots B_T^{(J)\frac{k_J-1}{2}-n_J+\delta}$$

since $\Sigma n_\alpha = \ell$. This can be rewritten as

$$T^{-J/2+\delta}\left(\frac{B_T^{(1)}}{B_T^{(2)}}\right)^{\frac{k_1-1}{2}}\left(\frac{B_T^{(2)}}{B_T^{(3)}}\right)^{\frac{k_1-1}{2}+\frac{k_2-1}{2}-n_2}$$

$$\dots\left(\frac{B_T^{(J-1)}}{B_T^{(J)}}\right)^{\frac{k_1-1}{2}+\dots+\frac{k_{J-1}-1}{2}-n_2-\dots-n_J} B_T^{(J)\frac{k_1-1}{2}+\dots+\frac{k_J-1}{2}-n_2-\dots-n_J+\delta} .$$

Since

$$(4.42)\qquad \frac{k_1+\dots+k_\alpha-1}{2} \geq n_2+\dots+n_\alpha$$

for $\alpha < J$, the above is less than or equal to

(4.43)

$$T^{-J/2+\delta}\left(\frac{B_T^{(1)}}{B_T^{(2)}}\right)^{\frac{k_1-1}{2}}\left(\frac{B_T^{(2)}}{B_T^{(3)}}\right)^{-\frac{1}{2}}\left(\frac{B_T^{(3)}}{B_T^{(4)}}\right)^{-1}\cdots\left(\frac{B_T^{(J-1)}}{B_T^{(J)}}\right)^{-\frac{J-2}{2}} B_T^{(J)\,-\frac{J}{2}+\delta}$$

$$= T^{-J/2+\delta}\left(\frac{B_T^{(1)}}{B_T^{(2)}}\right)^{\frac{k_1-1}{2}} B_T^{(2)\,-\frac{1}{2}}\cdots B_T^{(J-1)\,-\frac{1}{2}}\, B_T^{(J)\,\delta-1} = o(1)$$

as $T \to \infty$.

REFERENCES


[1] Blanc-Lapierre, A. and Fortet, R. (1953). Théorie des Fonctions Aléatories. Masson, Paris.

[2] Brillinger, D. R. (1965). An introduction to polyspectra. Ann. Math. Statist. 36, 1351-1374.

[3] Doob, J. L. (1953). Stochastic Processes. Wiley, New York.

[4] Kolmogorov, A. N. (1960). On the $\Phi^{(n)}$ classes of Fortet and Blanc-Lapierre. Theor. Prob. Appl., 5, 337.

[5] Leonov, V. P. and Shiryaev, A. N. (1959). On a method of calculation of semi-invariants. Theor. Prob. Appl., 4, 319-329.

[6] Leonov, V. P. (1964). Some Applications of Higher-Order Semi-invariants to the Theory of Stationary Random Processes (In Russian), Izdatelstvo Nauka, Moscow.

[7] Parzen, E. (1956). On consistent estimates of the spectrum of a stationary time series. Ann. Math. Statist., 28, 329-348.

[8] Rosenblatt, M. (1966). Remarks on higher order spectra. Symposium on Multivariate Analysis. (To be published).





[9] Rosenblatt, M. and Van Ness, J. W. (1965). Estimation of the bispectrum. Ann. Math. Statist., 36, 1120-1136.

[10] Shiryaev, A. N. (1960). Some problems in the spectral theory of higher-order moments, I. Theor. Prob. Appl., 5, 265-284.

[11] Shiryaev, A. N. (1963). On conditions for the ergodicity of stationary processes in terms of higher order moments. Theor. Prob. Appl., 8, 436-439.

[12] Sinai, Ya. G. (1963). On higher order spectral measures of ergodic stationary processes. Theor. Prob. Appl., 8, 429-435.

[13] Van Ness, J. W. (1966). Asymptotic normality of bispectral estimates. (To appear in Ann. Math. Statist.)


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