Asymptotic Normality of Finite Fourier Transforms of Stationary Generalized Processes*

DAVID R. BRILLINGER

The University of California, Berkeley

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This paper indicates a mixing condition under which a net of Fourier transforms, of a stationary generalized process over an abelian locally compact group, has a limiting normal distribution.

1. INTRODUCTION

Finite Fourier transforms of stationary mixing processes have been shown to be asymptotically normal in quite a variety of circumstances. The case of a time series $X(t)$ with t in R is considered in, for example, Leonov and Shiryaev [12], Picinbono [16], Rosenblatt [19], Rozanov [21]. The case of t in Z is considered in Hannan [8, Chap.IV], in Hannan and Thomson [9], in Brillinger [5]. The case of t in R^p is investigated in Brillinger [2] and in Pichard [17]. Morettin [13] considers the case of t in a non-compact locally compact second countable group. In extensions of another sort Brillinger [3] indicates conditions leading to asymptotic normality in the case of stationary random measures on R and Brillinger [4] in the case of stationary random Schwartz distributions with t in R^p. This paper provides a central limit theorem for Fourier transforms of stationary random Schwartz–Bruhat distributions over a locally compact abelian group G. In the general case nets of Fourier transforms are shown to be asymptotically normal. When G is σ-compact the nets become sequences. The Fourier transforms need not be asymptotically normal. Rosenblatt [20] derives a non-Gaussian limit for the transform of a process with long range dependence.

In this work G denotes a locally compact abelian group, Λ denotes its dual. Haar measures on these two groups are denoted by dg and $d\lambda$, respectively. The group operation is denoted by "+". Characters of the groups are denoted by $\langle g, \lambda \rangle$ for g in G and λ in Λ. The Fourier transform of a function ψ in $L_1(G)$ is defined by

$$\Psi(\lambda) = \int \langle g, -\lambda \rangle \, \psi(g) \, dg. \tag{1.1}$$

If Ψ belongs to $L_1(\Lambda)$ one has the inversion formula

$$\psi(g) = \int \langle g, \lambda \rangle \, \Psi(\lambda) \, d\lambda. \tag{1.2}$$

In the case of Euclidean space, the infinitely differentiable functions of rapid decrease provide a setting "par excellence" (Schwartz [22, p. 104]) for harmonic analysis. Similar functions exist in the group case and may be defined as follows (Osborne [15]).

First one defines $A(G)$ the space of functions whose L_∞ norm decreases rapidly off powers of a compact set. Specifically $\psi \in A(G)$ if there exists a compact set $C(\psi) \subset G$ such that for each positive integer n there is a constant M_n such that for each integer $k \geqslant 1$

$$\| \psi |_{G - C(\psi)^k} \|_\infty \leqslant M_n k^{-n}. \tag{1.3}$$

One has, for example, $A(G) \subset L_p(G)$, $p \geqslant 1$, $L_\infty(G) \cdot A(G) \subset A(G)$ and $A(G) * A(G) \subset A(G)$. $A(G)$ is translation invariant.

The Schwartz–Bruhat space, $S(G)$, of rapidly decreasing test functions consists of functions $\psi \in A(G)$ with Fourier transforms $\Psi \in A(\Lambda)$. The space has a topology arising from the inequalities defining $A(G)$ and $A(\Lambda)$. One has $S(G) \subset A(G)$, $A(G) * S(G) \subset S(G)$, $S(G) \cdot S(G) \subset S(G)$. $S(G)$ is translation invariant.

A Schwartz–Bruhat tempered distribution is a continuous linear functional on $S(G)$. Such a distribution $x(\psi)$, $\psi \in S(G)$ has a Fourier transform $X(\Psi)$, $\Psi \in S(\Lambda)$, whose (inverse) Fourier transform is $x(\psi)$ in turn.

A final space that will be made use of is $D(G) = S(G) \cap K(G)$, where $K(G)$ denotes the space of continuous functions of compact support on G. One introduces a finer topology on $D(G)$ than that of $S(G)$. (See the references below.) One has $D(G) * D(G) \subset D(G)$ and $D(G) \cdot D(G) \subset D(G)$, and $D(G)$ is dense in $S(G)$. A Schwartz–Bruhat (ordinary) distribution is a continuous linear functional on $D(G)$. A tempered distribution is an (ordinary) distribution which is continuous with respect to the $S(G)$-topology.

The above concepts are discussed in Bruhat [6], Wawrzynczyk [25], Osborne [15], Argabright and Gil de Lampard [1].

2. STATIONARY GENERALIZED PROCESSES

This paper is concerned with random Schwartz–Bruhat distributions. These are defined as mean–square continuous (in the $D(G)$-topology) random linear functionals on $D(G)$, whose values are equivalence classes of complex-valued random variables of second-order defined on some probability space $(\Omega, \mathcal{Q}, P)$ and which form a Hilbert space $L_2(\Omega)$. (See Ponomarenko [18].) In the stationary case the covariance function of such a process has a spectral representation

$$\text{cov}\{x(\psi_1), x(\psi_2)\} = \int \Psi_1(\lambda) \, \overline{\Psi_2(\lambda)} \, F_2(d\lambda). \tag{2.1}$$

F_2 has a non-negative measure on Λ and the process itself has a spectral representation

$$x(\psi) = \int \Psi(\lambda) \, Z(d\lambda) = X(\Psi) \tag{2.2}$$

with Z a stochastic measure satisfying $\text{cov}\{Z(A), Z(B)\} = F_2(A \cap B)$. The domain of x may be expanded to $\psi \in S(G)$ and to ψ with $\Psi \in L_2(F_2)$. In the case that the process x is real $Z(d\lambda) = \overline{Z(-d\lambda)}$. (See Ponomarenko [18].)

The above spectral representations were developed by Kampé de Fériet [11] for the case of a stationary ordinary process over a group G and by Niemi [14] for the case of a stationary random measure on a group G and by Cramér [7] originally.

A mixing condition will be required in order to develop the central limit theorems of this paper. It is,

ASSUMPTION I. x is a real stationary generalized process. For $k = 1, 2, \ldots$ the cumulant of order $k + 1$ of $x(\psi)$ is assumed to exist and be given by

$$\int \cdots \int \Psi(\lambda_1) \cdots \Psi(\lambda_k) \, \overline{\Psi(\lambda_1 + \cdots + \lambda_k)} \, f_{k+1}(\lambda_1, \ldots, \lambda_k) \, d\lambda_1 \cdots d\lambda_k \tag{2.3}$$

with

$$\text{vrai sup} \, |\Psi(\lambda_1) \cdots \Psi(\lambda_k) \, \Psi(\lambda_1 + \cdots + \lambda_k) \, f_{k+1}(\lambda_1, \ldots, \lambda_k)| < \infty \tag{2.4}$$

for all $\Psi \in S(\Lambda)$.

The functions f_{k+1} are called the cumulant spectra of the process x. They are given in terms of the stochastic measure Z by

$$\text{cum}\{Z(d\lambda_1),...,Z(d\lambda_{k+1})\}$$
$$= \delta(\lambda_1 + \cdots + \lambda_{k+1}) f_{k+1}(\lambda_1,...,\lambda_k) \, d\lambda_1 \cdots d\lambda_{k+1}, \qquad (2.5)$$

where δ is the Dirac delta function.

Making an observation on the process x will be viewed as being able to compute $x(\psi)$ for all ψ in $D(G)$ having support within a given compact set T, (the domain of observation). The sample sum will be viewed as the value $x(\psi)$ with ψ near 1 throughout T. The finite Fourier transform will be taken to be

$$x(\langle g, -\omega \rangle \psi) = \int \Psi(\omega - \lambda) Z(d\lambda) = X(\Psi(\omega - \cdot)) \qquad (2.6)$$

for such a ψ. The function ψ will need to be in $D(G)$ and will be called a taper.

In the case that G is not compact and that T is large the finite Fourier transform (2.5) might be expected to be approximately normally distributed when the process x is mixing. The notion that T is large will be formalized by basing the Fourier transform on a net of tapers $\{\psi_\alpha\}$ such that $\lim \psi_\alpha(g) = 1$ (pointwise) for all $g \in G$. (In the case that G is σ-compact the net may be taken to be a sequence.) It will be seen that nets of tapers exist such that the finite Fourier transform is asymptotically normal.

3. The Asymptotic Normality

The following regularity conditions will be required in connection with the net of tapers.

Assumption II. $\{\Psi_\alpha\}$ is a net of functions in $S(\Lambda)$ with the property that Ψ_α may be written $\Psi_\alpha = \Phi_\alpha \Psi$ with Φ_α, $\Psi \in S(\Lambda)$, $\Psi(0) = 1$

$$\int |\Phi_\alpha|^2 = 1, \qquad \lim \int_{\Lambda/U} |\Phi_\alpha|^2 = 0 \qquad (3.1)$$

for U any neighborhood of 0 in Λ and

$$\lim \int |\Phi_\alpha| = 0. \qquad (3.2)$$

In Appendix 1 of the paper such a net will be constructed in the case of non-compact G with the further properties that $\psi_\alpha \in D(G)$ and that $\psi_\alpha(g)$ is proportional to a function with limit $= 1$ for $g \in G$. In the case of σ-compact G, the net may be taken to be a sequence.

The principal result of the paper may now be stated.

THEOREM. *Let the stationary generalized process x satisfy Assumption I and have mean 0. Suppose further that the second-order spectrum $f_2(\lambda)$ is continuous at the distinct points $\lambda = \omega_1, \ldots, \omega_J$ and does not vanish at those points. Let the net (or sequence) $\{\Psi_\alpha\}$ satisfy Assumption II. Then the finite Fourier transforms $X(\Psi_\alpha(\omega_j - \cdot))$, $j = 1, \ldots, J$ are asymptotically independent normals with zero means and variances $f_2(\omega_j)$, $j = 1, \ldots, J$, respectively.*

Proof. The argument proceeds by showing that the cumulants of order greater than 2 tend to 0 and that those of order 2 tend to the values indicated. Because the normal distribution is determined by its cumulants (those of order greater than 2 vanish), and because the cumulants exist and tend to 0 as indicated, the corresponding net of probability measures is tight and asymptotic normality follows. The argument is carried through for the univariate case. The joint normality argument may be reduced to this case. Using abbreviated notation, one has for cumulants of order $k + 1$

$$|\operatorname{cum}\{X(\Psi_\alpha), \ldots, X(\Psi_\alpha)\}|$$

$$= \left| \int \Psi_\alpha(\lambda_1) \cdots \Psi_\alpha(\lambda_k) \overline{\Psi_\alpha(\lambda_1 + \cdots + \lambda_k)} f_{k+1}(\lambda_1, \ldots, \lambda_k) d\lambda_1 \cdots d\lambda_k \right|$$

from (2.3)

$$\leqslant M \int |\Phi_\alpha(\lambda_1) \cdots \Phi_\alpha(\lambda_k) \Phi_\alpha(\lambda_1 + \cdots + \lambda_k)| d\lambda_1 \cdots d\lambda_k \text{ from (2.4)}$$

$$\leqslant M \left(\int |\Phi_\alpha(\lambda)|^{(k+1)/k} d\lambda \right)^k \text{ from Appendix 2.}$$

Now $(\int |\Phi|^{(k+1)/k})^k \leqslant \int |\Phi|^2 (\int |\Phi|)^{k-1}$ by Holder's Inequality and so the cumulant indicated tends to 0 from (3.2) in the case $k > 1$. (Use is also made here of the fact that the space $S(G)$ is translation invariant.)

In the case $k = 1$

$$\operatorname{var} X(\Psi_\alpha(\omega - \cdot)) = \int |\Psi_\alpha(\omega - \lambda)|^2 f_2(\lambda) d\lambda$$

$$= \int |\Phi_\alpha(\omega - \lambda)|^2 |\Psi(\omega - \lambda)|^2 f_2(\lambda) d\lambda$$

tending to $f_2(\omega)$ at points of continuity of f_2 in view of (3.1), the approximate identity nature of the net $|\Phi_\alpha|^2$. In the covariance case one has

$$|\mathrm{cov}\{X(\Psi_\alpha(\omega_1 - \cdot)), X(\Psi_\alpha(\omega_2 - \cdot))\}|$$

$$= \left| \int \Psi_\alpha(\omega_1 - \lambda) \, \overline{\Psi_\alpha(\omega_2 - \lambda)} \, f_2(\lambda) \, d\lambda \right|$$

$$\leqslant M \int |\Phi_\alpha(\omega_1 - \lambda) \, \Phi_\alpha(\omega_2 - \lambda)| \, d\lambda$$

from (2.4). This last integral may be seen to tend to 0 for $\omega_1 \neq \omega_2$ by splitting it up into an integral over a neighborhood of ω_1 not containing ω_2 and a remainder and then using Schwarz's Inequality separately on each integral.

This completes the proof of the Theorem.

4. Extensions

The theorem extends to the case of vector-valued processes by means of the same arguments. This extension includes the complex-valued case.

Appendix 1

This Appendix demonstrates the existence of a net of functions, $\Psi_\alpha \in S(\Lambda)$ satisfying Assumption II.

Simon [24] and Hewitt and Ross [10, p. 298] construct nets, $\Gamma_\alpha \in L_1(\Lambda)$, such that: (i) $\Gamma_\alpha \geqslant 0$, $\int \Gamma_\alpha(\lambda) \, d\lambda = 1$; (ii) $0 \leqslant \gamma_\alpha(g) \leqslant 1$, $\gamma_\alpha \in K(G)$; (iii) given U a neighborhood of 0 in Λ and $\beta > 0$, there is an α_0 such that for $\alpha > \alpha_0$ one has $\int_U \Gamma_\alpha(\lambda) \, d\lambda > 1 - \beta$ and $\Gamma_\alpha(\lambda) < \beta$ for $\lambda \in \Lambda/U$; (iv) $\lim \gamma_\alpha(g) = 1$ uniformly on compact subsets of G; and (v) the net becomes a sequence for G σ-compact.

Bruhat [6] demonstrates the existence of functions $\theta \in D(G)$ such that $\theta \geqslant 0$ and $\int \theta(g) \, dg = 1$ and one has $|\Theta| \leqslant 1$.

Now set

$$\phi_\alpha = \gamma_\alpha * \theta \Big/ \left(\int (\gamma_\alpha * \theta)^2 \right)^{1/2}, \qquad \Phi_\alpha = \Gamma_\alpha \Theta \Big/ \left(\int (\Gamma_\alpha \Theta)^2 \right)^{1/2}.$$

One sees that $\gamma_\alpha * \theta(g) \geqslant 0$ and tends to $\int \theta(g) \, dg = 1$. This implies that $\int \Gamma_\alpha^2 \Theta^2$ tends to ∞ for non-compact G. Further, from (iii),

$$\int_{\Lambda/U} \Gamma_\alpha^2 \Theta^2 \leqslant \beta(1-\beta)$$

giving (3.1). Next, $\int |\Gamma_\alpha \Theta| \leqslant \int \Gamma_\alpha \leqslant 1$ and (3.2) follows. The construction is completed by taking $\Psi = \Theta$.

Appendix 2

Lemma. *Let k be a positive integer and Φ in $L_p(\Lambda)$ with $p = (k + 1)/k$. Then*

$$\int \cdots \int |\Phi(\lambda_1) \cdots \Phi(\lambda_k) \Phi(\lambda_1 + \cdots + \lambda_k)| \, d\lambda_1 \cdots d\lambda_k \leqslant \left(\int |\Phi(\lambda)|^p \, d\lambda \right)^k.$$

Proof. The result follows from Holder's Inequality, namely,

(a) $\|fg\|_1 \leqslant \|f\|_p \|g\|_q$, $1/p + 1/q = 1$, Young's inequality (Segal and Kunze [23, p. 204], namely,

(b) $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$, $1/r = 1/p + 1/q - 1 \geqslant 0$ and Fubini. Define $H_0(\lambda) = \Phi(\lambda)$,

$$H_{j+1}(\lambda) = \int |\Phi(\lambda + \mu) H_j(\mu)| \, d\mu$$

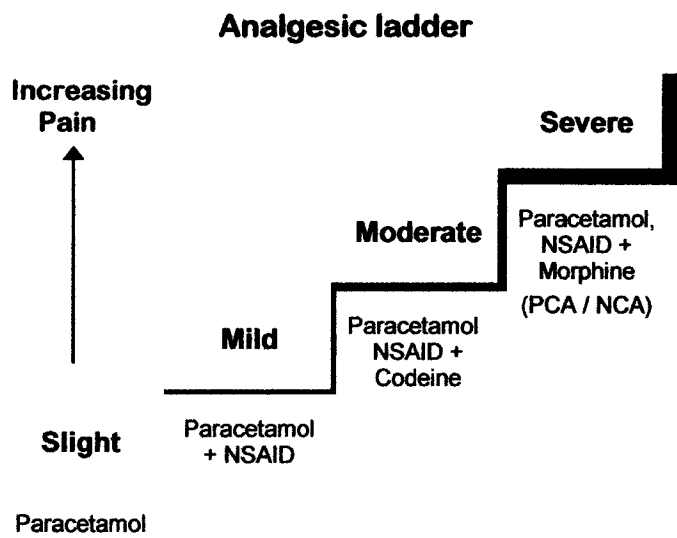and note that if the integral exists it is given by $\int |\Phi(\lambda) H_{k-1}(\lambda)| \, d\lambda$. Now,

$$\int |\Phi(\lambda) H_{k-1}(\lambda)| \, d\lambda \leqslant \|\Phi\|_p \|H_{k-1}\|_{k+1}$$

$$\|H_{k-1}\|_{k+1} \leqslant \|\Phi\|_p \|H_{k-2}\|_{(k+1)/2} \leqslant \cdots \leqslant \|\Phi\|_p^{k-1} \|H_0\|_{(k+1)/k}$$

$$= \|\Phi\|_p^k.$$

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Analgesic ladder

Increasing Pain ↑

Severe

Moderate

Paracetamol,
NSAID +
Morphine

(PCA / NCA)

Mild

Paracetamol
NSAID +
Codeine

Slight

Paracetamol
+ NSAID

Paracetamol

NB: ■ *Check 'British National Formulary' for contraindications / interactions / precautions etc*

PCA / NCA / Epidural patients

0	**No pain** *	
1 - 3	**Mild pain** *	
	NCA	- give bolus (10 mins before activity)
	PCA	- encourage bolus (10 mins before activity)
4 - 7	**Moderate pain** *	
	NCA	- give bolus
	PCA	- encourage bolus
	EPIDURAL	- contact Pain Service
8 - 10	**Severe pain** *	
	NCA	⎤
	PCA	⎬ - contact Pain Service
	EPIDURAL	⎦

* Ensure supplementary analgesia is given (paracetamol + an NSAID if appropriate)
NB: No codeine