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Estimation of uncertainties in eigenspectral estimates from decaying geophysical time series

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Summary. The response of many dynamical systems to an impulse is a linear combination of decaying cosines. The frequencies of the cosines have generally been estimated in geophysics by periodogram analysis and little formal indication of uncertainty has been provided. This work presents an estimation procedure by the methods of complex demodulation and non-linear regression that specifically incorporates in the basic model the decaying aspect of the cosines (periodogram analysis does not). The use of plots of the instantaneous phase as a function of time is shown to greatly enhance resolution. Expressions for the variances of eigenfrequencies, amplitudes, phases and damping constants Q are derived by non-linear least-squares. The results are illustrated, for the problem of the free oscillations of the Earth, by computations with the record made at Trieste of the Chilean earthquake of 1960 May 22. Sample values are periods and standard errors of 737.79 ± 0.13 s, 506.25 ± 0.13 s and 429.60 ± 0.14 s for ${}_0T_8$, ${}_0T_{13}$ and ${}_0T_{16}$ with Q values and standard errors of 200 ± 14, 230 ± 28 and 215 ± 30, respectively.

Introduction

A basic need in the measurement of terrestrial eigenspectra is a general algorithm for simultaneously estimating eigenfrequencies, amplitudes, phases and damping coefficients. This paper provides such a method, formulated in a statistical context so that variances of each estimate can also be obtained. The method also has wider applications.

From the beginning of work on the Earth's free vibrations, the emphasis has been on estimation of the spectral eigenfrequencies (Derr 1969; Buland & Gilbert 1978), but few estimates have been accompanied by statistical uncertainties. This requirement is important because independent frequency estimates have been seen to differ by up to 0.5 per cent on occasion (e.g., 2 s for ${}_0T_{14}$, ${}_0T_{17}$) and it is difficult to know how to combine the separate estimates.

Many fewer measurements are available of the actual ground displacements in each eigenvibration (Nowroozi 1974), partly because some key recording instruments were not calibrated for impulse response, but also because some methods of spectrum estimation used could not provide the true amplitudes. New work on terrestrial eigenvibrations is stressing not only measurements of the ground amplitudes but also the damping of amplitudes. As emphasized by several authors (Jobert & Roult 1976; Anderson & Hart 1978), even the most recent estimates of the damping constant (usually given in seismology as the specific dissipation constant Q) show considerable scatter and indicate the great difficulty of precise measurements of the amplitude decay rate. Further, there are questions of whether Q depends on frequency. Progress clearly depends upon the more systematic use of statistical analysis of the time series (Bolt & Brillinger 1975).

The procedures and formula developed in this paper were motivated by the problem of spectral estimation of damped terrestrial eigenvibrations. In particular, the computer programs were tested on the time series obtained by the long-period pendulums at the Grotta Gigante, Trieste, following the 1960 Chile earthquake (Bolt & Marussi 1962). These data have provided some of the best estimates of the gravest torsional eigenfrequencies to the present time. It is hardly necessary to point out, however, that the methodology developed is of a general nature and is applicable to a wide class of geophysical time series.

Our procedure depends heavily on the ability of complex demodulation (Tukey 1961) not only to locate as precise a value of an eigenfrequency as the data permit, but often to allow an assessment of whether difficulties in resolution are arising from such physical causes as multiple energy sources or splitting of peaks due to Earth inhomogeneities and rotation. We investigate especially the use of the instantaneous phase spectrum for decisions on resolution. This is a sensitive method that seems to have received little use in the analysis of geophysical periodicities previously. We set out an informative way of comparison between demodulate estimates of the amplitudes, frequencies and damping factors of the oscillations and with estimates obtained by the technique of non-linear regression (see Draper & Smith 1966). The latter technique allows the relative uncertainties between individual calculated eigenfrequencies to be estimated. The former gives a way to select the most closely resolved modes.

The model

The impulse response, $s(t)$, of a wide variety of stable geophysical, mechanical and electromagnetic linear systems with finite dissipation is a linear combination of decaying cosine waves,

$$s(t;\theta)=\sum_{k=1}^{K}\alpha_k \exp\{-\beta_k t\}\cos\{\gamma_k t+\delta_k\},\qquad t\geqslant 0, \tag{1}$$

where $\theta=\{\alpha_k,\beta_k,\gamma_k,\delta_k, k=1,\ldots,K\}$ with $\alpha_k,\beta_k,\gamma_k>0$, $0\leqslant\delta_k<2\pi$ and γ_k distinct. (See Lamb 1920, pp. 230–239; Whittaker 1944, pp. 230–234; Lancaster 1966, Chapter 9.) The γ_k are the eigenfrequencies of the system. The β_k determine the rate of decay of the oscillations and are often redefined as

$$\beta_k=\gamma_k/(2Q_k) \tag{2}$$

in terms of Q_k damping factors.

A traditional means of estimating the γ_k of equation (1) in geophysics has been the searching for peaks in the periodograms, or smoothed periodograms, calculated from the geophysical time series (see e.g. Zadro & Caputo 1968; Dziewonski & Gilbert 1972). The usual numerical procedure has been to calculate the amplitude Fourier spectrum only, using an FFT algorithm. A less usual method involves the fitting of a long autoregressive scheme to the digital record (Burg 1972; Bolt & Currie 1975). None of these estimation procedures have taken specific note of the presence of the damping factor β_k in equation (1), even though, as Dahlen (1978) has lately shown, the concept of damped sinusoids is valuable in theoretical discussions of terrestrial eigenvibrations and multiplets. Also, as mentioned above, their use has generally not been accompanied by the provision of formal indications of the statistical variability of the estimates.

The suggested approach is multi-stage. Assume that one of the traditional methods has been used to determine frequencies that perhaps correspond to eigenvibrations. Then complex demodulation (discussed in the next section) is carried out at the determined frequencies. Examination of the results of complex demodulation suggests whether an individual frequency is reasonable and allows initial estimation of a precise value for the frequency, decay, phase and amplitude. Finally non-linear regression, based on the Fourier transform values in the neighbourhood of a given frequency, is carried out in order to determine final estimates of the spectral parameters and their standard errors.

Complex demodulation

Given a record $X(t)$, $t = 1, \ldots, T$, the complex demodulate at frequency λ of that record is the time series $W(t, \lambda)$, $t = 1, 2, \ldots$, that results from low-pass filtering the series $X(t) \exp\{-i\lambda t\}$. The complex demodulate $W(t, \lambda)$ will be much smoother than the original time series. The technique is described in detail in Bingham, Godfrey & Tukey (1967), Brillinger (1975, p. 33), Bloomfield (1976, Chapter 6), for example.

In the present application, suppose the low-pass filter adopted has impulse response $b(t)$, transfer function $B(\lambda)$ with sufficiently small bandwidth and suppose λ is near an eigenfrequency γ_k. The demodulate may be written

$$W(t, \lambda) = \Sigma b(t-u) X(u) \exp\{-i\lambda t\}. \tag{3}$$

For the signal $s(t, \theta)$ of equation (1), the result of demodulating is

$$Z(t, \lambda) \simeq \tfrac{1}{2} B(0) \alpha_k \exp\{-\beta_k t\} \exp\{i(\gamma_k - \lambda)t + i\delta_k\}. \tag{4}$$

Standardize the low-pass filter by $B(0) = 2$ as we may. Then, from the complex demodulate, one sees the following forms for the instantaneous phase function

$$\arg Z(t, \lambda) \simeq (\gamma_k - \lambda)t + \delta_k \tag{5}$$

and for the logarithm of the instantaneous amplitude function

$$\log_e |Z(t, \lambda)| \simeq -\beta_k t + \log_e \alpha_k. \tag{6}$$

It follows that plots of $\arg W(t, \lambda)$ and $\log |W(t, \lambda)|$ against t can provide evidence of the presence of a damped periodicity in a time series of interest. Indeed, successive variations of the demodulate frequency λ lead to parameter trajectories from which α, β, γ and δ can be estimated in some optimal sense. If the plots (see Figs 1–4) of $\arg W(t, \lambda)$ and $\log_e |W(t, \lambda)|$ are made nearly linear over the record duration T, especially where the signal amplitude is large, the damped vibration is close to the adopted model. If the plot of $W(t, \lambda)$ is erratic, there is a suggestion that the record is just noise. If the plots have regular non-linear behaviour, there is some violation of the basic simple model, perhaps beating between signal and noise harmonics with nearly equal frequencies, perhaps the injection of new energy into the system by applied forces (perhaps an aftershock arrived), perhaps there is time dependent dispersion.

The slope of the logarithm of the instantaneous amplitude curve gives an estimate of the decay constant β_k; the intercept gives the log instantaneous amplitude of the oscillation at the beginning of the movement. Similarly, the intercept of the instantaneous phase plot yields the relative phase of the oscillation. In addition, it should be noted that some idea of the uncertainty of these estimates is given by the variation of the complex demodulate curves about the fitted straight lines over the selected time interval.

Non-linear regression

Consider first, data generated by a model

$$y_j = f_j(\theta) + e_j,$$

$j = 1, \dots, J$ where the y_j are observed, where the $f_j(\theta)$ are known except for the K-dimensional parameter θ, and where the e_j are unobserved, uncorrelated random errors with mean 0 and common variance σ^2. The least squares estimate of θ is the value providing the minimum of the expression

$$\sum_{j=1}^{J} |y_j - f_j(\theta)|^2.$$

Suppose that the function $f_j(\theta)$ is differentiable with derivatives

$$g_{jk}(\theta) = \frac{\partial f_j(\theta)}{\partial \theta_k}$$

$k = 1, \dots, K$. Collect the y_j together into the J-vector $\mathbf{y}$, the $f_j(\theta)$ into the J-vector $\mathbf{f}(\theta)$ and the $g_{jk}(\theta)$ into the $J \times K$ matrix $\mathbf{g}(\theta)$. One means of determining an extreme value of θ is through the Gauss–Newton iteration procedure

$$\theta^{n+1} = \theta^n + [\mathbf{g}(\theta^n)^* \mathbf{g}(\theta^n)]^{-1} \mathbf{g}(\theta^n)^* [\mathbf{y} - \mathbf{f}(\theta)] \tag{7}$$

$n = 0, 1, 2, \dots$, having started with some initial value θ^0. (Other procedures are described in Chambers (1973).) Under regularity conditions this estimate will be approximately normal with mean θ and covariance matrix that may be estimated by

$$[\mathbf{g}(\theta^n)^* \mathbf{g}(\theta^n)]^{-1} \sum_j |y_j - f_j(\theta^n)|^2/(J - K) \tag{8}$$

(see Jennrich 1969).

In the present case, where the noise is not uncorrelated and K is very large, it seems appropriate to modify the above approach as follows. Suppose

$$X(t) = s(t, \theta) + \epsilon(t) \qquad t = 1, \dots, T$$

where $s(t, \theta)$ is given by equation (1) and $\epsilon(t)$ is a stationary noise series with mean 0 and power spectrum $f_{\epsilon\epsilon}(\lambda)$. Define

$$\Delta^T(\lambda) = \sum_{t=1}^{T} \exp\{-i\lambda t\}$$

$$d_x^T(\lambda) = \sum_{t=1}^{T} X(t) \exp\{-i\lambda t\},$$

$0 \leqslant \lambda \leqslant 2\pi$, with similar definitions for $d_s^T(\lambda), d_\epsilon^T(\lambda)$. By Parseval's formula

$$\sum_{t=1}^{T} |X(t) - s(t,\theta)|^2 = T^{-1} \sum_{j=0}^{T-1} \left| d_x^T\left(\frac{2\pi j}{T}\right) - d_s^T\left(\frac{2\pi j}{T}\right) \right|^2. \tag{9}$$

Minimizing the left-hand side of this expression is equivalent to minimizing the right-hand side. Now $d_s^T(\lambda)$ is a sum of the terms

$$\sum_{t=1}^{T} \alpha_k \exp\{-\beta_k t\} \cos\{\gamma_k t + \delta_k\} \exp\{-i\lambda t\} = b_k \Delta^T(\lambda - \chi_k) + \bar{b}_k \Delta^T(\lambda + \chi_k),$$

$k = 1, \ldots, K$, where $b_k = \frac{1}{2}\alpha_k \exp\{i\delta_k\}, \chi_k = \gamma_k + i\beta_k$. By inspection, the term in $\Delta^T(\lambda - \chi_k)$ has appreciable magnitude only for λ near γ_k. This means that the minimum of equation (9) may be obtained approximately by simultaneously minimizing the expressions

$$\sum_{I_k} \left| d_x^T\left(\frac{2\pi j}{T}\right) - b_k \Delta^T\left(\frac{2\pi j}{T} - \chi_k\right) \right|^2 \tag{10}$$

where $I_1, \ldots, I_k$ are disjoint frequency intervals making up the interval $[0, \pi]$, and $2\pi j/T, \lambda_k$ belong to I_k. In addition, for $2\pi j/T$ in I_k,

$$d_\epsilon^T\left(\frac{2\pi j}{T}\right) \simeq d_x^T\left(\frac{2\pi j}{T}\right) - b_k \Delta^T\left(\frac{2\pi j}{T} - \chi_k\right)$$

are approximately independent complex normal variates with zero mean and common variance $2\pi T f_{\epsilon\epsilon}(\lambda)$. (See Brillinger 1975, Theorem 4.4.1. This approximation seems to work very well in practice, *ibid.*)

We proceed by computing the $d_x^T(2\pi j/T)$ using a fast Fourier transform algorithm, identifying the intervals I_k from the periodogram of the record $X(t)$ and then estimating b_k, χ_k by minimizing expression (10) using a Gauss–Newton iteration procedure. The covariance matrix of these estimates may be estimated by an expression analogous to equation (8).

If we think of β_k as fractions of T, say $\beta_k = \phi_k/T$ as seems reasonable in practice (for otherwise the signal $s(t, \theta)$ would quickly become a negligible part of $X(t)$ as t increases) then if $\hat{\alpha}_k, \hat{\phi}_k, \hat{\gamma}_k, \hat{\delta}_k$ denote the least squares estimates it can be shown, by direct extension of the arguments of Hannan (1973) that

$$\begin{aligned} \operatorname{var} \hat{\alpha}_k &\sim T^{-1} 4\pi f_{\epsilon\epsilon}(\gamma_k) I_2(\phi_k) J(\phi_k)^{-1} \\ \operatorname{var} \hat{\phi}_k &\sim T^{-1} 4\pi f_{\epsilon\epsilon}(\gamma_k) \alpha_k^{-2} I_0(\phi_k) J(\phi_k)^{-1} \\ \operatorname{var} \hat{\gamma}_k &\sim T^{-3} 4\pi f_{\epsilon\epsilon}(\gamma_k) \alpha_k^{-2} I_0(\phi_k) J(\phi_k)^{-1} \\ \operatorname{cov}\{\hat{\alpha}_k, \hat{\phi}_k\} &\sim T^{-1} 4\pi f_{\epsilon\epsilon}(\gamma_k) \alpha_k^{-1} I_1(\phi_k) J(\phi_k)^{-1} \\ \operatorname{cov}\{\hat{\gamma}_k, \hat{\delta}_k\} &\sim T^{-2} 4\pi f_{\epsilon\epsilon}(\gamma_k) \alpha_k^{-2} I_1(\phi_k) J(\phi_k)^{-1}, \end{aligned} \tag{11}$$

$k = 1, \ldots, K$ with all other covariances asymptotically negligible, where

$$I_l(\phi) = \int_0^1 u^l \exp\{-2\phi u\}\, du$$

$l = 0, 1, 2$

$$J(\phi) = I_0(\phi) I_2(\phi) - I_1(\phi)^2.$$

Expressions analogous to those of equation (11) are derived for the case of $\beta_k = 0, k = 1, \ldots, K$ in Whittle (1954), Walker (1971), Hannan (1973).

In the case that a separate record of the noise process, $\epsilon(t)$, is available, it may be used to estimate $f_{\epsilon\epsilon}(\gamma)$ directly and alternate estimates of the variances of interest may be constructed through the formulas of equation (11). This should be done whenever possible as it seems that the variance estimates should be more robust than those produced by the non-linear regression. We were unable to do this in the present case of the Trieste data.

Numerical results

The steps outlined in the paper were applied to the Trieste record of the 1960 May 22 Chilean earthquake. The data were digitized at a time interval of 2 min and tides were removed. The number of data points was 2548 points. The periodogram of the record was examined for peaks. Demodulation was carried out at the peak frequencies (Bolt & Currie 1975). A representative selection of the results is described below. The coefficients $b(t)$ of equation (3) were taken proportional to $1 + \cos(\pi u/L)$ for $|u| < L$ and were 0 otherwise with $L = 200$. (For a lengthy stretch of data, it would have been advantageous to employ a fast Fourier transform in the computations (see Bingham *et al.* 1967)).

Consider first the demodulates for ${}_0T_4$ shown in Fig. 1. The instantaneous phase remains almost constant until about 25 hr from the onset. This is followed by small variations in phase until almost 40 hr when the phase increases sharply, becoming erratic at about 50 hr. There is thus evidence that there is almost a pure harmonic, at about the demodulation frequency chosen, for at least 25 hr and perhaps for another 10. The behaviour of the instantaneous amplitude is consistent with the phase information; a straight line equation (6) might well be fitted to the first 25–35 hr and a decay rate β_k measured. After this time the amplitudes become erratic with large fluctuations which suggest the level of background noise has been reached. Some variation in the fitting of the line equation (6) even to the first part of the spectrum is, however, clearly permissible and numerical fits of straight lines indicate slopes corresponding to Q values between extremes of 300 and 400 are perhaps allowable. If a longer period of recording were used, however, Fig. 1 indicates that a lower decay rate might be calculated (i.e., a false high Q). Comparison with similar plots shows

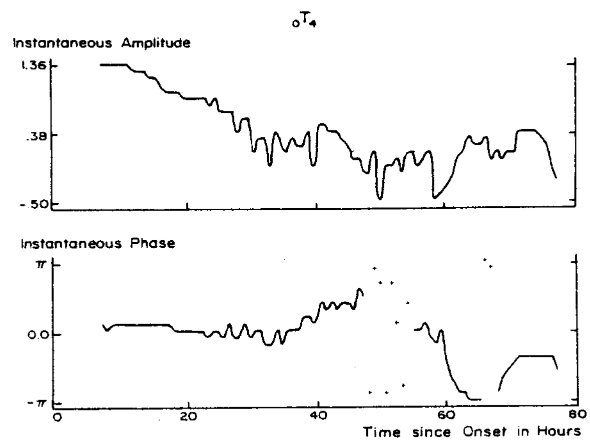

Figure 1. Results of complex demodulation for the mode ${}_0T_4$ at a demodulation period of 1303.15 s from the N–S horizontal Trieste record of the 1964 Chilean earthquake. The upper plot gives the $\log_{10}$ instantaneous ground amplitude as a function of time in hours from the beginning of the record. The lower plot shows the variation in instantaneous phase between $-\pi$ and π with time in hours.

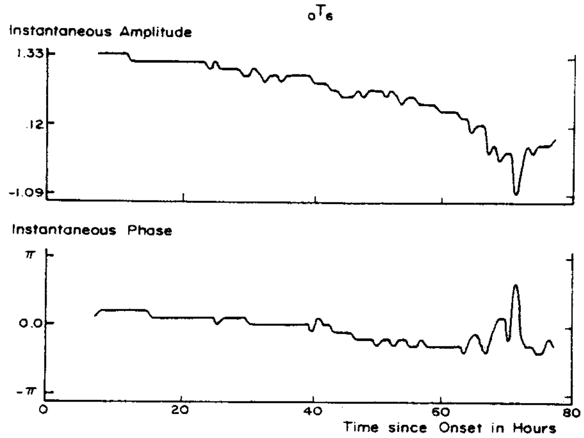

Figure 2. Complex demodulation for $_0T_6$ at a demodulation period of 925.65 s for the Trieste data.

that the $_0T_4$ mode gives one of the more stable instantaneous plots calculated from the Trieste data. In this regard it should be noted that this eigenvibration is well separated from neighbouring torsional and spheroidal modes (see Fig. 1, Bolt & Currie 1975) so that no interference is expected.

Now, consider the similarly isolated $_0T_6$ mode, demodulated in Fig. 2. Here there is even more stability of instantaneous phase and amplitude than for $_0T_4$. There is only a slight drift in phase over the first 65 hr. (This slight drift may indicate that the demodulation frequency adopted could be improved slightly.) The decay for $_0T_6$ is clearly similar to that for $_0T_4$ (note different vertical scales), and a straight line can be fitted to the instantaneous amplitude up to 60 hr with comparable precision. Overall, we would expect the estimate of the $_0T_6$ frequency to be more reliable than that of $_0T_4$.

In Fig. 3, the instantaneous spectra for the demodulates of the spheroidal mode $_0S_9$ are shown. In this case, apart from a hiatus near the beginning of the record, the phase is almost constant near -0.7π throughout the record. We thus have an assurance that we are measuring a single coherent decaying sinusoid throughout most of the recording. Further study of the instantaneous amplitude confirms this, although there is a more rapid decrease

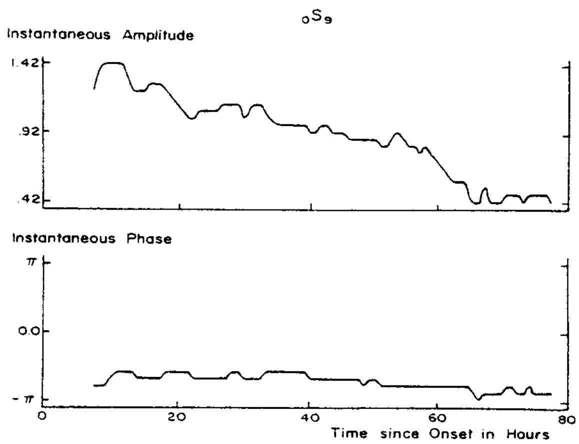

Figure 3. Complex demodulation for $_0S_9$ at a demodulation period of 634.90 s for the Trieste data.

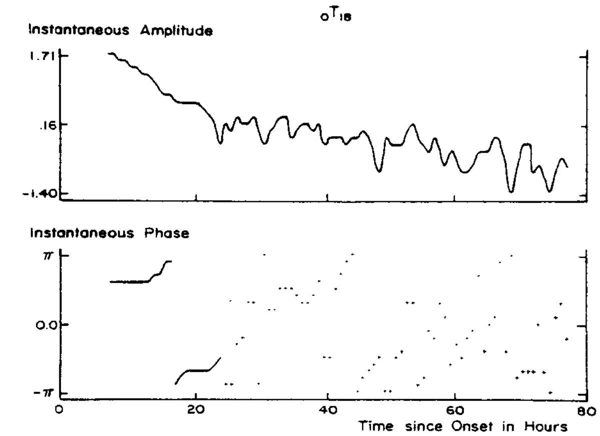

Figure 4. Complex demodulation for $_0T_{18}$ at a demodulation period of 391.45 s for the Trieste data.

in amplitude after about 60 hr. Some allowance for this change can be made in estimating the decay rate β_k. The observed Q is clearly significantly higher for $_0S_9$ than for $_0T_4$ and $_0T_6$ and appears moderately well resolved in the sense of the straight line fit equation (6). The explanation of the change in slope at 60 hr remains unknown, but presumably the effect of interfering signals (noise?) has become more important.

The fourth demodulate presented (Fig. 4) is an illustration of a mode for which the instantaneous phase plot detects major difficulties in resolution. Even 20 hr after the onset the phase angle begins to change rapidly and thereafter cannot be followed. (Note that phase moves continuously from the top to the bottom of the plot.) The conclusion is that only the first 20 hr of amplitudes should be used to estimate Q for this mode. A mean Q value of only about 190 is indicated by the slope of a fitted line. Thereafter the amplitude decays more slowly with large fluctuations. It is interesting that the spectral peak of $_0T_{18}$ is very near that of its neighbour $_0S_{17}$ and it is feasible that some cross modulation (or leakage) is occurring.

Table 1 gives the non-linear least squares estimates of the periods, Q-factors, and relative initial amplitudes and phases for the modes that complex demodulation suggested were truly present and were not multiple. The figures in brackets below give estimates of the corresponding standard errors. (The model was reparameterized to estimate eigenperiods, rather than frequencies, as these seem to be the more usual values discussed.)

The iteration scheme converged exceedingly rapidly. In the case of the modes indicated, the results presented are those obtained after 10 iterations. The case of split peaks might have been handled by fitting the sum of two decaying cosine waves within a frequency interval I_k.

Conclusions

The present paper demonstrates the advantages of the complex demodulation technique for the spectral analysis of geophysical time series composed of damped harmonic terms in the presence of noise. The discussion here focused on the critical problem of improving estimates of amplitudes, frequencies and Q values for the modes of damped eigenvibrations of the Earth. When comparison is possible (Bolt & Currie 1975), it is found that recent estimates of eigenfrequencies for some modes (e.g., $_0T_{14}$, $_0T_{17}$) differ by up to 0.5 per cent.

Table 1. Spectral estimates for the Trieste data (standard errors in parentheses).

Mode	Complex Demodulate Period (sec)	Q	Initial Amplitude (arbitrary units)	Phase (radians)
${}_0T_4$	1303.150	347.8	30.64	.216
	(.697)	(20.3)	(4.63)	(.151)
${}_0T_5$	1078.816	185.0	62.01	1.973
	(.371)	(23.6)	(8.07)	(.130)
${}_0S_6$	963.005	336.2	23.41	1.094
	(.267)	(62.7)	(4.30)	(.184)
${}_0T_6$	925.651	357.2	25.55	.160
	(.332)	(91.4)	(6.43)	(.252)
${}_0T_7$	818.377	124.8	127.38	-2.624
	(.427)	(16.2)	(17.35)	(.137)
${}_0T_8$	737.791	199.6	119.37	2.786
	(.129)	(14.0)	(8.74)	(.073)
${}_0S_8$	707.458	376.2	46.73	2.764
	(.202)	(81.0)	(10.10)	(.217)
${}_0S_6$	659.908	184.4	125.91	-2.921
	(.207)	(21.3)	(15.36)	(.122)
${}_0S_9$	634.087	658.1	23.18	-2.155
	(.086)	(117.3)	(3.92)	(.169)
${}_0T_{10}$	619.202	187.6	160.62	-3.049
	(.134)	(15.2)	(13.77)	(.086)
${}_0T_{13}$	506.251	230.3	140.77	.532
	(.134)	(28.1)	(17.88)	(.127)
${}_2S_8$	486.827	159.4	106.21	-.944
	(.118)	(12.3)	(9.33)	(.088)
${}_0T_{15}$	452.416	172.2	126.21	.040
	(.109)	(14.3)	(11.41)	(.091)
${}_0T_{16}$	429.604	214.5	155.35	2.940
	(.141)	(30.1)	(22.94)	(.148)
${}_0T_{18}$	391.447	188.3	119.32	1.929
	(.088)	(16.0)	(10.76)	(.091)
${}_0T_{20}$	359.399	248.7	84.15	-2.558
	(.041)	(14.2)	(5.05)	(.061)

However, it is difficult to combine the independent estimates because of the lack of comparable probability models. It is recommended that the present method be used so that pooling with appropriate weights can be made.

Studies of terrestrial eigenspectra have now advanced to the stage when analysis of a long record of free oscillations must provide more than a set of mean eigenfrequencies. Not only does the rotation and ellipticity of the Earth produce frequency multiplets about the central (degenerate) frequency, but lateral inhomogeneities split the peaks also. Earthquake sources of various types and at various locations generate at different seismographic stations different relative strengths in the multiplets. As well, variations in long-period noise spectra, activation of new sources, and rotation of the nodal lines, relative to the receiver, all produce fluctuations which complicate the meaning to be attached to a simple mean eigenfrequency estimate. It is demonstrated in this paper that the plots of the real and imaginary parts of the complex demodulates of each mode provide a powerful way to detect and explore such fluctuations. The eccentric behaviour is not 'swept under the rug' as occurs with most traditional methods. Already some progress in the geophysical interpretation of these plots has been made (Hansen 1978).

Formulae in the present paper enable programs to be written to compute relatively quickly the complex demodulates and, by non-linear least squares, variances of the spectral parameters. By repetition at successive steps in the demodulating frequency, the set of instantaneous amplitude and phase plots allows a decision to be made on the best eigenfrequency resolution available from the data and the quality of the damping factor Q and amplitude of ground motion that can be obtained. Clearly more experience with the method is needed before specific rules for decisions can be given.

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