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Computation and Interpretation of k-Th Order Spectra

I. Introduction and Summary of Results.

In this paper we turn to various aspects of the computation and interpretation of k-th order spectra. The effects of symmetries and the directly related problem of aliasing are discussed. In addition, we discuss the computational and algebraic properties of the estimate of a k-th order spectrum proposed in [3]. Brief discussions of models, pre-filtering, and band-pass filtering are included. We return to the classical series of sun-spot numbers [9] and compute estimates of the second, third and fourth order spectra.

II. Aspects of Symmetries.

A. Notation and Definitions. We will, to a certain extent, be commenting on results obtained in the accompanying paper on the asymptotic behavior of k-th order spectral estimates [3]. For this reason it will be convenient to adopt the notation employed there.

The process $X(t) = (X_a(t); a = 1, \ldots, r)$ is assumed to be a stationary r-vector valued process with real-valued components. All moments of the process are assumed to exist and equivalently all cumulant functions are assumed to exist. The moment functions

$$(2.1)\quad m_{a_1,\ldots,a_k}(t_1,\ldots,t_k) = E X_{a_1}(t_1)\ldots X_{a_k}(t_k) = m_{a_1,\ldots,a_k}(t+t_1,\ldots,t+t_k)$$

and cumulant functions

$$(2.2)\quad c_{a_1,\ldots,a_k}(t_1,\ldots,t_k) = c(X_{a_1}(t_1),\ldots,X_{a_k}(t_k)) = c_{a_1,\ldots,a_k}(t+t_1,\ldots,t+t_k)$$

many k-th order spectral densities. Specifically there may be a k-th order spectral density corresponding to each partition of k into j conjugated entries and $k-j$ unconjugated entries. This contrasts strongly with the real-valued case in which one has one k-th order moment function and correspondingly one k-th order spectral density.

B. Aliasing. It is often assumed that the basic process $X(t)$ has a continuous time parameter t, $-\infty < t < \infty$,

$$X(t) = \int_{-\infty}^{\infty} \exp\{it\lambda\}\, dZ(\lambda) \quad (2.11)$$

However if the process is sampled at times $t = nh$, $n = 0, \pm 1, \ldots$, then one can write

$$X(nh) = \int_{-\infty}^{\infty} \exp\{inh\lambda\}\, dZ(\lambda) = \int_{-\pi/h}^{\pi/h} \exp\{inh\lambda\}\, dW(\lambda) \quad (2.12)$$

where

$$W(\lambda+\Delta) - W(\lambda) = \sum_k \{Z(\lambda+\Delta+\frac{2k\pi}{h}) - Z(\lambda+\frac{2k\pi}{h})\} \quad (2.13)$$

and $|\Delta| \leq \pi/h$. Now if $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ is a k-th order spectral density for $X(t)$, then

$$f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)\delta(\sum_1^k \lambda_j)\, d\lambda_1 \ldots d\lambda_k = c(dZ_{a_1}(\lambda_1), \ldots, dZ_{a_k}(\lambda_k)) . \quad (2.14)$$

The corresponding k-th order spectral density $g_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ of the discretely sampled process $X(nh)$ can be related rather simply to $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$. Specifically,

(2.15)

$$g_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)\, \eta(h\sum_1^k \lambda_j)\, d\lambda_1 \ldots d\lambda_k = c(dW_{a_1}(\lambda_1), \ldots, dW_{a_k}(\lambda_k))$$

$$= \sum_{j_\alpha} f_{a_1,\ldots,a_k}(\lambda_1 + \frac{2\pi j_1}{h}, \ldots, \lambda_k + \frac{2\pi j_k}{h})\, \delta(\sum_{\alpha=1}^k (\lambda_\alpha + \frac{2\pi j_\alpha}{h}))\, d\lambda_1 \ldots d\lambda_k .$$

Under regularity conditions implying the validity of the multi-dimensional Poisson summation formula, we may conclude

$$g_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k) = \sum f_{a_1,\ldots,a_k}(\lambda_1 + \frac{2\pi j_1}{h}, \ldots, \lambda_k + \frac{2\pi j_k}{h}) \quad (2.16)$$

where $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$, and the summation extends over j_α such $\Sigma_{\alpha=1}^k(\lambda_\alpha + \frac{2\pi j_\alpha}{h}) = 0$. (Such a regularity condition is that there exist $A, \epsilon > 0$ with $|{}_k c_{a_1,\ldots,a_k}(t_1,\ldots,t_{k-1})| \leq A(1+\|t\|)^{-k+1-\epsilon}$ and $|{}_k f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_{k-1})| \leq A(1+\|\lambda\|)^{-k+1-\epsilon}$, $\|t\|^2 = \Sigma_1^{k-1} t_j^2$ and $\|\lambda\|^2 = \Sigma_1^{k-1}\lambda_j^2$.) For convenience in the remainder of our discussion we take $h = 1$. The basic space of wave numbers $(\lambda_1,\ldots,\lambda_k)$ in the k-th order discrete case is the direct product of k circles $(-\pi, \pi]$, which we shall call a k-torus. However the spectral mass in the k-th order case is located on the manifold $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$. This is a $(k-1)$-torus cut obliquely out of the k-torus. In the case $k = 2$, this is just the circle of points $(\lambda, -\lambda)$, $-\pi < \lambda \leq \pi$. It is usual to only use the first component to represent the typical point on the manifold. For $k > 2$, it is not clear that the analogous procedure is the completely natural way in which to represent the $(k-1)$-torus on which the k-th order spectral mass is located. However, it is not unreasonable and we shall consider the analogous procedure for $k = 3, 4$. We want all components of a typical frequency vector to be less than or equal to π in absolute value.

The case $k = 3$ is first considered. We are giving a Euclidean representation to a manifold that is globally non-Euclidean. All vectors $(\lambda_1, \lambda_2, \lambda_3)$ with $-\pi < \lambda_j \leq \pi$ and $\Sigma\lambda_j \equiv 0 \pmod{2\pi}$ are to be accounted for. They will be represented on planes corresponding to $\lambda_1+\lambda_2+\lambda_3 = 0$, $\lambda_1+\lambda_2+\lambda_3 = 2\pi$, $\lambda_1+\lambda_2+\lambda_3 = -2\pi$. The points will be represented on these planes by their first two coordinates. Figure 1 represents the points on $\lambda_1+\lambda_2+\lambda_3 = 0$. The dotted lines represent the lines of symmetry $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_3$. Actually only half the points on Figure 1 are required, let us say on the right of $\lambda_1 = \lambda_3$, because of (2.10). In Figure 2, the triangle with vertices $(-\pi, 0, \pi)$, $(-\pi, -\pi, 0)$ and $(0, -\pi, -\pi)$ is the part of this 2-torus represented on $\lambda_1+\lambda_2+\lambda_3 = -2\pi$. In Figure 3, the triangle with vertices $(0, \pi, \pi)$, $(\pi, \pi, 0)$ and $(\pi, 0, \pi)$ is the part of the 2-torus on $\lambda_1+\lambda_2+\lambda_3 = 2\pi$. Since the triangle in Figure 2 is just the mirror image of the triangle in Figure 3 through the point $(0, 0, 0)$ condition (2.10) implies that we need only consider the triangle in Figure 3. We see that one need only consider the part of the hexagon in Figure 1 to the right of $\lambda_1 = \lambda_3$ together with the triangle in Figure 3. One can easily account for the further reduction of the region in the case that there are further symmetries due to the identification of some of the

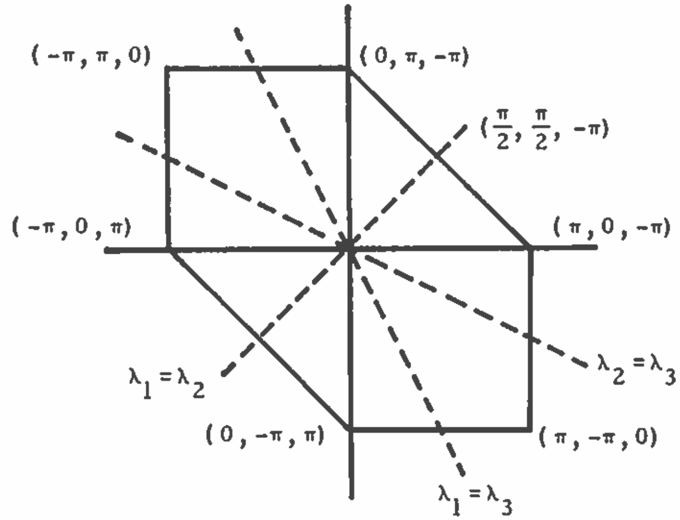


FIGURE 1

$\lambda_1+\lambda_2+\lambda_3 = 0$

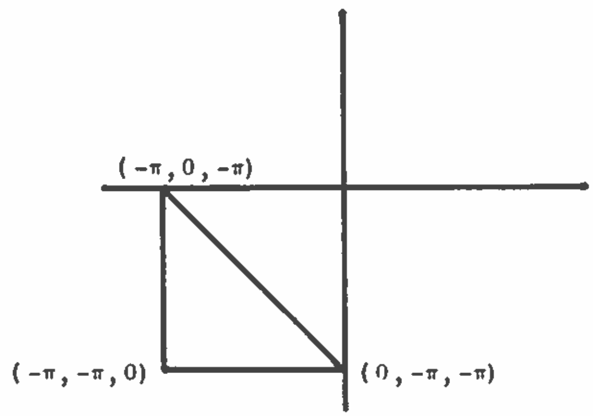


FIGURE 2

$\lambda_1+\lambda_2+\lambda_3 = -2\pi$

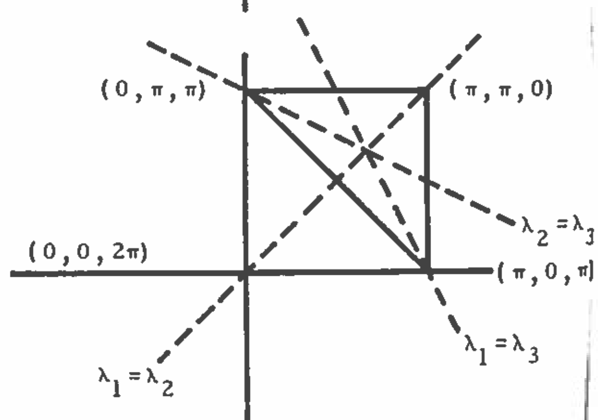


FIGURE 3

$\lambda_1+\lambda_2+\lambda_3 = 2\pi$

components whose 3-spectrum one is computing. Thus if $a_1 = a_2 = a$, we have

$$f_{a,a,b}(\lambda_1, \lambda_2, \lambda_3) = f_{a,a,b}(\lambda_2, \lambda_1, \lambda_3) . \tag{2.17}$$

This implies that one need only consider the section of the hexagon in Figure 1 that is a pentagon with vertices $(\frac{\pi}{2}, \frac{\pi}{2}, -\pi)$, $(\pi, 0, -\pi)$, $(\pi, -\pi, 0)$, $(\frac{\pi}{2}, -\pi, \frac{\pi}{2})$ and $(0, 0, 0)$ and the triangle in Figure 3 with vertices $(\pi, \pi, 0)$, $(\pi, 0, \pi)$ and $(\frac{\pi}{2}, \frac{\pi}{2}, \pi)$. In the case $a_1 = a_2 = a_3$ one has full symmetry and the region is reduced to the triangle with vertices $(0, 0, 0)$, $(\frac{\pi}{2}, \frac{\pi}{2}, -\pi)$, $(\pi, 0, -\pi)$ in the plane $\lambda_1+\lambda_2+\lambda_3 = 0$, together with the triangle with vertices $(\frac{\pi}{2}, \frac{\pi}{2}, \pi)$, $(\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$, $(\pi, 0, \pi)$ in the plane $\lambda_1+\lambda_2+\lambda_3 = 2\pi$. (See Figure 4).

We now give a less extended discussion in the case $k = 4$. The vectors $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ with $-\pi < \lambda_j \leq \pi$ and $\Sigma \lambda_j \equiv 0 \pmod{2\pi}$ are to be considered. These points can be represented on $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0, \pm 2\pi$ by their first three coordinates $\lambda_1, \lambda_2, \lambda_3$. We shall only describe the points on $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$ and $= 2\pi$. The points on $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = -2\pi$ are obtained from those on $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2\pi$ by reflection through $(0, 0, 0, 0)$ and hence are not required by condition (2.10). The basic region in $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 0$ is the regular octahedron given in Figure 5. Only half of this is required by virtue of (2.10) say the part on one side of $\lambda_1 = \lambda_4$ together with the pyramid in $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2\pi$ given in Figure 6. In the case of full symmetry, that is with $a_1 = a_2 = a_3 = a_4$ the basic domain can be reduced to the pyramid with vertices $(0, 0, 0, 0)$, $(\pi, \pi, -\pi, -\pi)$, $(\pi, 0, 0, -\pi)$, $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, -\pi)$ in $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ and the pyramid with vertices $(\pi, 0, 0, \pi)$, $(\pi, \pi, -\pi, \pi)$, $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \pi)$, $(\pi, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ in $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 2\pi$. (See Figure 7).

A discussion of aliasing, in the case of a single time series and for a 3-rd order spectral estimate, may be found in [8].

III. Computational Aspects.

A. The Proposed Estimate. In the previous paper, [3], we defined the finite Fourier transform,

$$d_a^{(T)}(\lambda) = \sum_{t=0}^{T-1} X_a(t) \exp\{-i\lambda t\} , \tag{3.1}$$

FIGURE 4

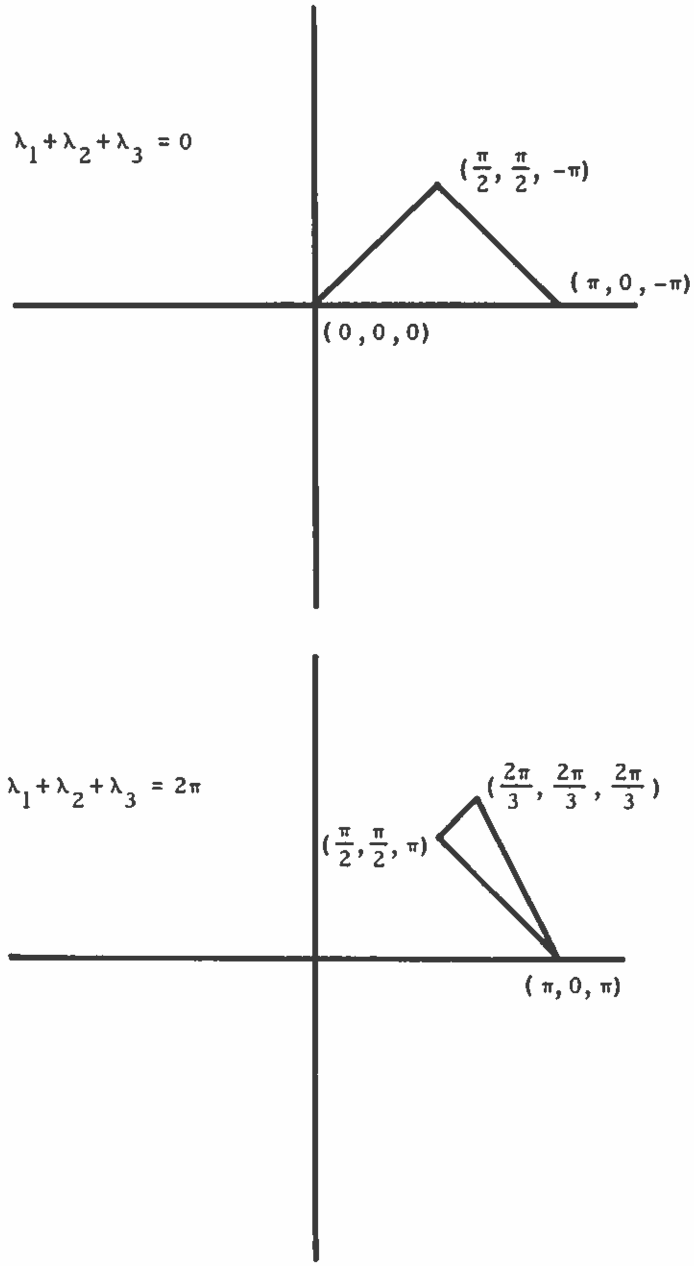


FIGURE 5

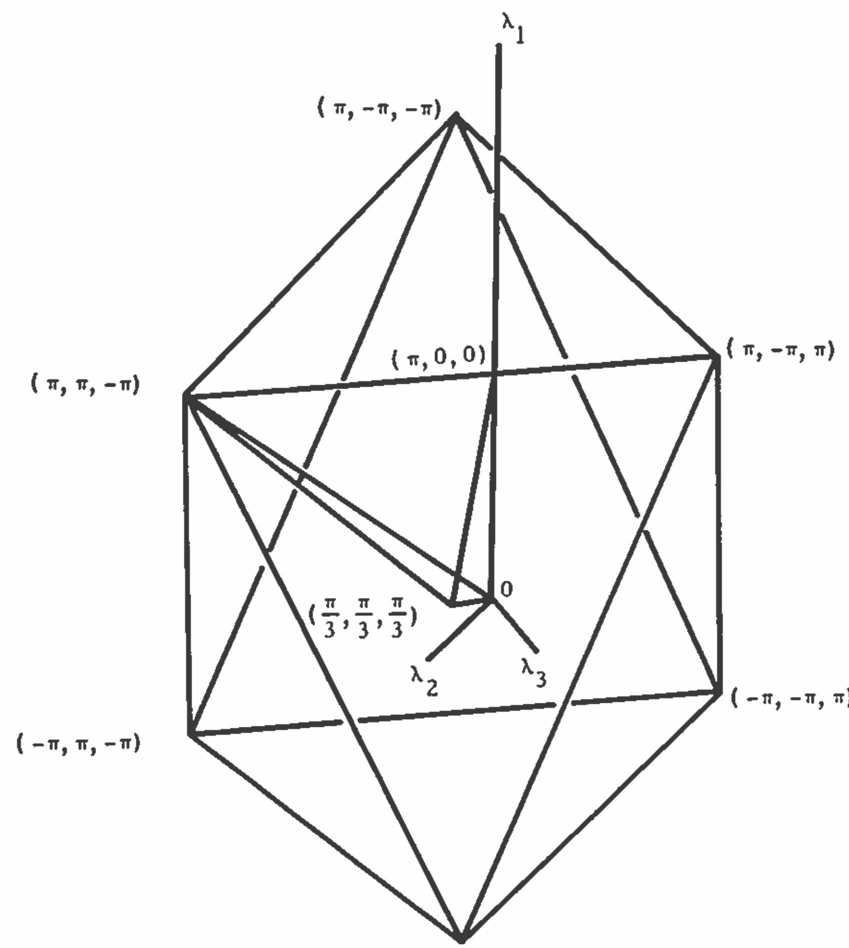


FIGURE 6

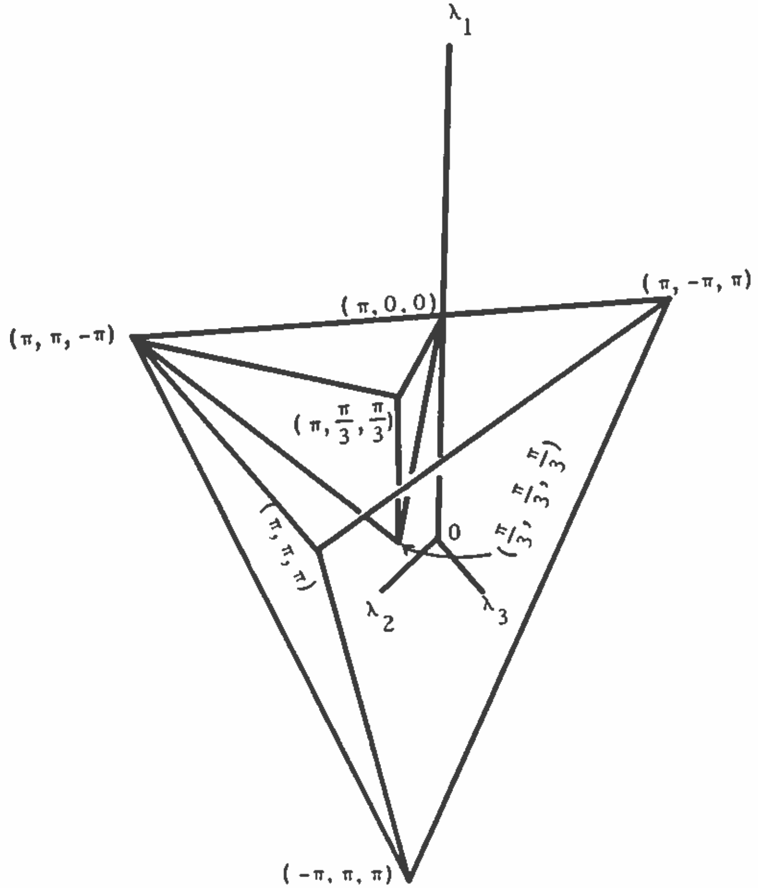
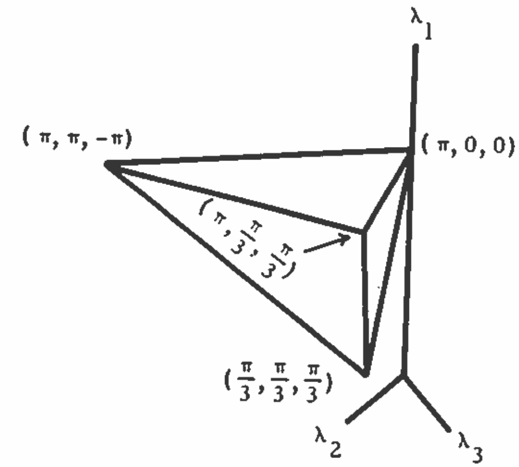
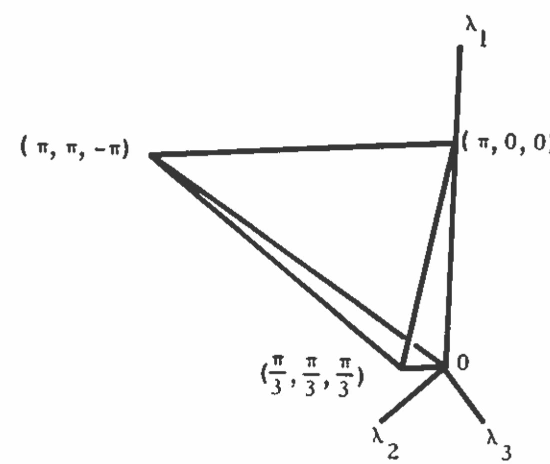


FIGURE 7





the k-th order periodogram,

$$I^{(T)}_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k) = (2\pi)^{-k+1}T^{-1}\prod_{j=1}^{k} d^{(T)}_{a_j}(\lambda_j) \tag{3.2}$$

and given the weight function $W(u_1,\ldots,u_k)$ on the plane $\Sigma_1^k u_j = 0$ with

$$W_T(u_1,\ldots,u_k) = B_T^{-k+1} W(B_T^{-1}u_1,\ldots,B_T^{-1}u_k) \tag{3.3}$$

$$\overline{W}_T(u_1,\ldots,u_k) = \sum W_T(u_1+2j_1\pi,\ldots,u_k+2j_k\pi) \tag{3.4}$$

the summation in (3.4) extending over $j_1,\ldots,j_k$ with $u_1+\ldots+u_k+2\pi(j_1+\ldots+j_k) = 0$, we defined the estimate,

$$\begin{aligned} f^{(T)}_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k) &= (2\pi)^{k-1}T^{-k+1}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty} W_T\left(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right) \\ &\quad\cdot I^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right) \\ &= (2\pi)^{k-1}T^{-k+1}\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1} \overline{W}_T\left(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right) \\ &\quad\cdot I^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right). \end{aligned} \tag{3.5}$$

In (3.5) we have $\Sigma_1^k\lambda_j \equiv 0 \pmod{2\pi}$ and $\Phi(u_1,\ldots,u_k)$ equals 1 if $\Sigma_1^k u_j \equiv 0 \pmod{2\pi}$, but $\Sigma_{j\epsilon J}u_j \not\equiv 0 \pmod{2\pi}$, where J is any nonvacuous proper subset of $1,\ldots,k$, and $\Phi(u_1,\ldots,u_k) = 0$ otherwise. We note that the estimate involves the expressions

$$\sum_{t=0}^{T-1} X_a(t)\exp\left\{-i\,\frac{2\pi st}{T}\right\}, \tag{3.6}$$

the discrete Fourier transform of the sequence $X_a(0),\ldots,X_a(T-1)$. If T is highly composite, this discrete Fourier transform may be calculated rapidly via the Fast Fourier Transform Algorithm (see [7].) This is a distinct advantage of the present technique of obtaining an estimate by a smoothing in the frequency domain, rather than the time domain. Other advantages include an easy handling of missing values and unequally spaced time points and an easy calculation of estimates with differing values of B_T.

If we define the associated circular process $\hat{X}(t)$ by $\hat{X}(t) = X(t)$ for $0 \le t \le T-1$ and the periodic extension, with period T, outside this range, we may prove[1],

Lemma 1. If

$$\hat{m}_{a_1,\ldots,a_k}(v_1,\ldots,v_k) = T^{-1}\sum_{t=0}^{T-1}\hat{X}_{a_1}(t+v_1)\ldots\hat{X}_{a_k}(t+v_k) \tag{3.7}$$

then

$$\hat{m}_{a_1,\ldots,a_k}(v_1,\ldots,v_k) = T^{-k}\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1}\exp\left\{\sum_1^k s_jv_j\right\} \cdot J^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right)\eta_{s_1+\ldots+s_k} \tag{3.8}$$

where

$$J^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right) = \prod_1^k d^{(T)}_{a_j}\left(\frac{2\pi s_j}{T}\right). \tag{3.9}$$

Also if

$$\hat{c}_{a_1,\ldots,a_k}(v_1,\ldots,v_k) = \sum_\nu (-1)^{p-1}(p-1)!\,\hat{m}_{a_{\nu_1}}(v_{\nu_1})\ldots\hat{m}_{a_{\nu_p}}(v_{\nu_p}) \tag{3.10}$$

the summation extending over all partitions $(\nu_1,\ldots,\nu_p)$ of the integers $(1,\ldots,k)$, then

$$\hat{c}_{a_1,\ldots,a_k}(v_1,\ldots,v_k) = T^{-k}\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1}\exp\left\{i\frac{2\pi}{T}(s_1v_1+\ldots+s_kv_k)\right\} \cdot J^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right) \tag{3.11}$$

[1] See Section VI for the proof of this result.

and

$$(3.12)\quad I^{(T)}_{a_1,\ldots,a_k}\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right)$$

$$=(2\pi)^{-k+1}\sum_{v_1=-T+1}^{T-1}\cdots\sum_{v_{k-1}=-T+1}^{T-1}\hat{c}_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1},0)\exp\left\{-i\sum_1^{k-1}v_j\frac{2\pi s_j}{T}\right\}.$$

We note from (3.5) and (3.12) that the proposed estimate may be written in the form

$$(3.13)\quad (2\pi)^{-k+1}\sum_{v_1=-T+1}^{T-1}\cdots\sum_{v_{k-1}=-T+1}^{T-1}\hat{w}_T(v_1,\ldots,v_{k-1})\hat{c}_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1},0)$$

$$\cdot\exp\left\{-i\sum_1^{k-1}v_j\lambda_j\right\}$$

where

$$(3.14)\quad \hat{w}_T(v_1,\ldots,v_{k-1})=(2\pi)^{k-1}T^{-k+1}\sum_{s_1=-\infty}^{\infty}\cdots\sum_{s_k=-\infty}^{\infty}W_T\left(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\right)$$

$$\cdot\exp\left\{i\sum_1^{k-1}v_j\left(\lambda_j-\frac{2\pi s_j}{T}\right)\right\}\eta_{s_1+\ldots+s_k}.$$

We see that the proposed estimate is the Fourier transform, with convergence factors, of the estimate of the joint cumulant derived from the associated circular process.

As a byproduct of Lemma 1, we note that we can derive the usual estimate of the joint cumulant using Fourier transform techniques as follows. Consider the series $X_a(0),\ldots,X_a(T-1),0,\ldots,0$ where T zeros have been added. Up to a factor of 2, the circular moment estimate of this series is the same as the product moment estimate of the original series. Taking note of (3.8) we see that we can derive the cumulant estimate for $|v_j|\leq T-1$ as

(3.15)

$$c^{(T)}_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1})=(2\pi)^{-k}\sum_{s_1=0}^{2T-1}\cdots\sum_{s_k=0}^{2T-1}\exp\left\{-i\frac{\pi}{T}(s_1v_1+\ldots+s_{k-1}v_{k-1})\right\}$$

$$\cdot J_{a_1,\ldots,a_k}\left(\frac{\pi s_1}{T},\ldots,\frac{\pi s_k}{T}\right)\sum_{\nu}(-1)^{p-1}(p-1)!\,2^p\,\eta_{\tilde{s}_{\nu_1}}\cdots\eta_{\tilde{s}_{\nu_p}}$$

where $\tilde{s}_\nu$ denotes the sum of the s's with subscripts in ν and the summation is carried out over all partitions $(\nu_1,\ldots,\nu_p)$ of $(1,\ldots,k)$. If T is highly composite, this may prove a much faster technique of calculating $c^{(T)}_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1})$.

B. Tapering and Prefiltering. Let us investigate the behavior of the proposed estimate when the data available are sections of discrete cosinusoids. Suppose $X_a(t)=b_a^{(\ell)}\exp\{i\theta_\ell t\}$. Then

$$(3.16)\quad d_a^{(T)}(\lambda)=\Delta_T(\lambda-\theta_\ell)b_a^{(\ell)}$$

where

$$(3.17)\quad \Delta_T(\lambda)=\sum_{t=0}^{T-1}\exp\{-i\lambda t\}.$$

We see that,

$$(3.18)\quad I^{(T)}_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)=(2\pi)^{-k+1}T^{-1}\prod_1^k\Delta_T(\lambda_j-\theta_{\ell_j})\,b_{a_j}^{(\ell_j)}.$$

Remembering that $\Sigma_1^k\lambda_j\equiv 0\ (\mathrm{mod}\ 2\pi)$, we see that we will have peaks at $(\lambda_1,\ldots,\lambda_k)=(\theta_{\ell_1}+2\pi j_1,\ldots,\theta_{\ell_k}+2\pi j_k)$ if $\Sigma_1^k\theta_{\ell_j}\equiv 0\ (\mathrm{mod}\ 2\pi)$. We can now rewrite the estimate given by (3.5) as

$$(3.19)\quad f^{(T)}_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$$

$$=T^{-k}\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1}\overline{W_T}\left(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\right)\prod_1^k\Delta_T\left(\frac{2\pi s_j}{T}-\theta_{\ell_j}\right)b_{a_j}^{(\ell_j)}.$$

We see that we will have peaks at $(\lambda_1,\ldots,\lambda_k)=(\theta_{\ell_1}+2\pi j_1,\ldots,\theta_{\ell_k}+2\pi j_k)$ if $\Sigma_1^k\theta_{\ell_j}\equiv 0\ (\mathrm{mod}\ 2\pi)$, but the sum over no subset of the θ's is congruent to zero (mod 2π). This last is undoubtedly a warning that if the sum over some subset is in fact congruent to zero (mod 2π), we would be better off carrying out a lower order analysis to locate that subset. If

$$(3.20)\quad X_a(t)=\sum_\ell b_a^{(\ell)}\exp\{i\theta_\ell t\},$$

then

$$I^{(T)}_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k) = (2\pi)^{-k+1}T^{-1}\sum_{\ell_1,\ldots,\ell_k}\prod_1^k \Delta_T(\lambda_j-\theta_{\ell_j})\, b_{a_j}^{(\ell_j)}, \quad (3.21)$$

and we see that the estimate will have peaks at $(\lambda_1,\ldots,\lambda_k) = (\theta_{\ell_1}+2\pi j_1,\ldots,\theta_{\ell_k}+2\pi j_k)$ whenever $\Sigma_1^k\theta_{\ell_j} \equiv 0 \pmod{2\pi}$, but no subset of the θ's has a sum congruent to zero (mod 2π).

Let us consider (3.19) in greater detail. $\Delta_T(\lambda)$ has the form,

$$\exp\{-i\tfrac{1}{2}\lambda(T-1)\}\sin\tfrac{1}{2}T\lambda / \sin\tfrac{1}{2}\lambda \quad (3.22)$$

and, as is well known, $\sin\frac{1}{2}T\lambda/\sin\frac{1}{2}\lambda$ has quite pronounced ripples extending over a broad range of frequencies. These ripples may well distort the estimate seriously. If, instead of (3.1), we form

$$d_a^{(T)}(\lambda) = \sum_{t=0}^{T-1} f_T(t)\, X_a(t)\exp\{-i\lambda t\} \quad (3.23)$$

then $\Delta_T(\lambda)$ is replaced by

$$F_T(\lambda) = \sum_{t=0}^{T-1} f_T(t)\exp\{-i\lambda t\} \quad (3.24)$$

and we see that we may reduce the rippling by a judicious choice of an $F_T(\lambda)$. The use of an $f_T(t)$ is called tapering in view of the form it generally takes in the time domain. (This idea is due to J. W. Tukey.) Typically one takes $f_T(t)$ to be $f(t/(T-1))$ where $f(u)$ is concentrated in the interval $0 \le u \le 1$, symmetric about $u = \frac{1}{2}$, of value 1 near $u = \frac{1}{2}$ and of value 0 near $u = 0$ or 1. Not unreasonable choices of functions $f(u)$ include a triangle function and the Tukey-Hamming weights $f(u) = \frac{1}{2}(1-\cos 2\pi u)$. Untapered estimates correspond to $f(u) = 1$.

An alternative technique of reducing the ripple effect is to operate directly with the $d_a^{(T)}(\frac{2\pi s}{T})$. In fact Tukey-Hamming weights correspond to forming

$$-\tfrac{1}{4}d_a^{(T)}\left(\tfrac{2\pi(s-1)}{T}\right)+\tfrac{1}{2}d_a^{(T)}\left(\tfrac{2\pi s}{T}\right)-\tfrac{1}{4}d_a^{(T)}\left(\tfrac{2\pi(s+1)}{T}\right). \quad (3.25)$$

Tapering is one means of improving an estimate; a second is prefiltering. In Theorem 2 of [3], we saw that if $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ was not flat, then a bias appeared in the estimate. Prefiltering is an operation whose purpose is to flatten $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ prior to analysis. The simplest form of prefiltering generally carried out is the subtraction of the mean. We note that in effect this is carried out by the insertion of Φ in (3.5) since the values of $d_a^{(T)}(\frac{2\pi s}{T})$, $s \ne 0, \pm T, \pm 2T, \ldots$ are unaffected by whether or not the mean has been subtracted, and Φ has the effect of causing the avoidance of $s = 0, \pm T, \pm 2T, \ldots$. We may carry this form of prefiltering a step further. The values of $d_a^{(T)}(\frac{2\pi s}{T})$, $s \ne n$, $n \pm T$, $n \pm 2T, \ldots$ are unaffected by whether or not a multiple of $\exp i2\pi nt/T$ has been subtracted. We see that if the values for $s = n$, $n \pm T$, $n \pm 2T, \ldots$ are avoided in the smoothing of the k-th order periodogram, then we experience the same effect as if we had prefiltered the series by fitting a cosinusoid $\exp\{i2\pi nt/T\}$ and then proceeded to analyse the residuals. The notion of avoiding certain points in periodogram smoothing was advanced independently by M. S. Bartlett at the London Meeting of Statisticians, September 1966.

Another common means of prefiltering a series, prior to estimating a second order spectrum, is to apply a linear time-invariant filter to the series. This method will also make sense in the k-th order situation, for if filters with transfer functions $H_{a_j}(\lambda)$ are applied to the $X_{a_j}(t)$, the k-th order spectrum of the resulting series is

$$H_{a_1}(\lambda_1)\ldots H_{a_k}(\lambda_k) f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k) \quad (3.26)$$

where as usual $\Sigma_1^k\lambda_j \equiv 0 \pmod{2\pi}$. We see that we may flatten the given k-th order spectrum by a judicious choice of the $H_{a_j}(\lambda)$.

(3.26) also gives us an indication of how the effect of aliasing might possibly be reduced. From (2.12) or (2.16) we see that if we have sampled a continuous-time time series at a sampling interval of h and if in the original series frequencies λ such that $|\lambda| \ge \pi/h$ are not present then aliasing does not occur. From the continuous analog of (3.26) we see that we may reduce the effect of frequencies λ, $|\lambda| \ge \pi/h$ by the application of suitable low-pass filters. We conclude that prior to digitizing a continuous-time time series at interval h, we should prefilter the series to attenuate frequency components at frequencies $\ge \pi/h$.

C. Band-pass Filters and Their Dangers. If the H's in (3.26) have the approximate form

$$H_{a_j}(\lambda) = \begin{cases} 1 & \omega_j - \frac{1}{2}\Delta\omega_j < \lambda < \omega_j + \frac{1}{2}\Delta\omega_j \\ 0 & \text{otherwise}^2 \end{cases} \quad (3.27)$$

with $\Sigma\omega_j \equiv 0 \pmod{2\pi}$, we see that after the application of the filters we have series whose k-th order spectrum is effectively concentrated at $(\omega_1, \ldots, \omega_k)$. In the 2-nd order situation, we know that if we have a narrow band process we may estimate the power in the band by estimating the variance of the process. In an analogous manner we would expect that for a process whose k-th order spectrum is concentrated in the neighborhood of a single point, the value of the k-th order spectrum at that point may be estimated by an estimate of the k-th order cumulant $c_{a_1,\ldots,a_k}(0,\ldots,0)$. From the expression

$$(3.28) \quad c_{a_1,\ldots,a_k}(t_1,\ldots,t_k)$$

$$= \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\{i\sum_1^k t_j\lambda_j\} f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)\,\eta(\sum_1^k \lambda_j)\,d\lambda_1\cdots d\lambda_k$$

we see that if the k-th order spectrum is concentrated as above

$$c_{a_1,\ldots,a_k}(0,\ldots,0) \approx f_{a_1,\ldots,a_k}(\omega_1,\ldots,\omega_k)\Delta\omega_1\cdots\Delta\omega_k \;. \quad (3.29)$$

If we denote the filtered series by $Y_{a_j}(t)$, and suppose that the filters each have bandwidth B_T, then we see that we are led to estimate $f_{a_1,\ldots,a_k}(\omega_1,\ldots,\omega_k)$ by

$$B_T^{-k+1}\sum_\nu(-1)^{p-1}(p-1)!\, n_{a_{\nu_1}}^{(T)}\cdots n_{a_{\nu_p}}^{(T)} \quad (3.30)$$

where the summation extends over all partitions $(\nu_1,\ldots,\nu_p)$ of $(1,\ldots,k)$, $a_\nu = (a_{i_1},\ldots,a_{i_j})$ if $\nu = (i_1,\ldots,i_j)$ and

$$n_{b_1,\ldots,b_j}^{(T)} = T^{-1}\sum_{t=0}^{T-1} Y_{b_1}(t)\cdots Y_{b_j}(t) \;. \quad (3.31)$$

[2] We note that series resulting from the application of such filters will be complex-valued.

A difficulty that may arise in the application of this estimation technique is that $n_{b_1,\ldots,b_j}^{(T)}$ given by (3.41) is essentially of order B_T^{j-1} and so the typical term of (3.40) is of order B_T^{-p+1}. In other words in the formation of (3.40) we will be adding and subtracting quantities of higher orders of magnitude than the quantity of interest. This technique, therefore, appears to be appropriate only under very special conditions.

D. Other Statistics. There are a variety of statistics that one may wish to calculate in addition to a k-th order spectral estimate. These include estimates of the modulus and argument of the k-th order spectrum, estimates of a standardized k-th order spectrum, $k \geq 3$,

$$\frac{f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)}{\prod_1^k\{f_{a_j,a_j}(\lambda_j,-\lambda_j)\}^{1/2}} \quad (3.32)$$

and the various coefficients given in [2]. We can derive these estimates by substitution of estimates of the various spectra. If we do this in the case of (3.32), using the same B_T in the 2-nd and k-th order estimates, we see from Theorems 3 and 4 of [3] that the asymptotic variance of the resulting estimate takes the form

$$2\pi B_T^{-k+1}T^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W^2(\beta_1,\ldots,\beta_k)\,\delta(\sum_1^k\beta_j)\,d\beta_1\cdots d\beta_k \quad (3.33)$$

off proper submanifolds of the principal manifold. (We note that if W has compact support, this asymptotic variance is minimized subject to $\int W = 1$ by setting W equal to a constant on its support and 0 elsewhere. Alternatively if W is expandable in terms of multidimensional Hermite functions, $H_{i_1,\ldots,i_j}(\beta_1,\ldots,\beta_j)$, one is led to take

$$W(\beta_1,\ldots,\beta_{k-1},-\beta_1-\cdots-\beta_{k-1}) = H_{0,\ldots,0}(\beta_1,\ldots,\beta_{k-1}) \;.$$

We note however that it may be desirable for W to have certain prespecified symmetry properties. In addition, note that (3.33) does not involve population parameters.) In the next section we will use this result to provide an indication of a basis on which to choose the B_T for spectral estimates of differing orders. A form equivalent to (3.33) is

$$(3.34)\qquad 2\pi T^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W_T^2(\beta_1,\ldots,\beta_k)\,\delta\Big(\sum_1^k \beta_j\Big)\,d\beta_1\cdots d\beta_k\ .$$

Direct substitution of estimates of the spectra involved is not the only means available for the construction of estimates. Suppose one wishes to estimate $\arg[f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)]$. Plausible estimates include

$$(3.35)\qquad \sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1} \overline{W_T}\Big(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\Big)\Phi\Big(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\Big)$$
$$\cdot\arg\Big[I^{(T)}_{a_1,\ldots,a_k}\Big(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\Big)\Big]$$

and

$$(3.36)\qquad \arg\sum_{s_1=0}^{T-1}\cdots\sum_{s_k=0}^{T-1} \overline{W_T}\Big(\lambda_1-\frac{2\pi s_1}{T},\ldots,\lambda_k-\frac{2\pi s_k}{T}\Big)\Phi\Big(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\Big)$$
$$\cdot\exp\Big\{i\arg\Big[I^{(T)}_{a_1,\ldots,a_k}\Big(\frac{2\pi s_1}{T},\ldots,\frac{2\pi s_k}{T}\Big)\Big]\Big\}\ .$$

(These expressions are related to expressions proposed by J. W. Tukey in a different situation.) Preliminary calculations have indicated that (3.36) does well in certain respects.

E. The Choice of the B_T. Let us suppose that we are considering a succession of estimates of the form of (3.5) of cumulant spectra of orders $k_1 \le k_2 \le \ldots \le k_J$ where the estimates involve scale factors $B_T^{(1)},\ldots,B_T^{(J)}$ and weight functions $W^{(1)},\ldots,W^{(J)}$. Intuitively one expects to have to increase the B_T as the order of the spectra increase in order to maintain reasonable stability in the estimate. (3.33) provides one means of deciding upon a satisfactory rate of increase. It indicates that in order to retain the same asymptotic variance of the estimates of the standardized spectra, $k \ge 3$, one should hold

$$(3.37)\qquad \{B_T^{(j)}\}^{-k_j+1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} W^{(j)}(\beta_1,\ldots,\beta_{k_j})\,\delta\Big(\sum_1^{k_j}\beta_\alpha\Big)\,d\beta_1\cdots d\beta_{k_j}$$

fixed for $j = 1,\ldots,J$. If the standardized spectrum in the 2-nd order case is considered to be $\log f_{a_1,a_1}(\lambda_1,\lambda_2)$, then this result continues

to hold for $k = 2$. If a 2-nd order B_T is now taken as base, we see that we are requiring the scale factor to increase proportionately with a root of the 2-nd order scale factor. If, for a specific example, we take

$$(3.38)\qquad W_T(\beta_1,\ldots,\beta_{k-1},-\beta_1-\ldots-\beta_{k-1}) = (B_T^{(k)})^{-k+1}\quad\text{for}\quad |\beta_j| \le \tfrac{1}{2}B_T^{(k)}$$

then (3.34) takes the form

$$(3.39)\qquad 2\pi(B_T^{(k)})^{-k+1}T^{-1}$$

and given $B_T = B_T^{(2)}$ we are led to set

$$(3.40)\qquad B_T^{(k)} = B_T^{1/(k-1)}\ .$$

IV. Interpretation of k-Th Order Spectra.

A. The Assumptions. At various stages in [3] and the present paper we set down a requirement of the form

$$(4.1)\qquad \sum_{v_1,\ldots,v_{k-1}=-\infty}^{\infty} |v_j|^{\ell}\,|c_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1},0)| < \infty$$

for $j = 1,\ldots,k-1$ and some integer $\ell \ge 0$ where $c_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1},0)$ denotes the joint cumulant of $X_{a_1}(t+v_1),\ldots,X_{a_{k-1}}(t+v_{k-1}),X_{a_k}(t)$. This assumption implied that the measure $dF_{a_1,\ldots,a_k}(\omega_1,\ldots,\omega_k)$ of (2.12) of [3] took on a particularly simple form. If instead we consider (2.10) of [3], we see that if we made an assumption of the form of (4.1) for the product moments or central product moments, then the measure $dG_{a_1,\ldots,a_k}(\omega_1,\ldots,\omega_k)$ would now take on a simple form. Mathematically it is certainly possible to find processes satisfying this latter sort of assumption and not the former. (Consider $X(t) = \xi$, for all t, where ξ is a random variable with $\mu_1(\xi) = 0$, $\mu_5(\xi) = 0 = \mu_5'(\xi)$, where $\mu_k(\xi) = E\xi^k$ and $\mu_k'(\xi) = E(\xi-\mu_1(\xi))^k$, but $\kappa_5(\xi) \ne 0$ and look at the 5-th order spectra.) We see that the choice of (4.1) relates to a judgement concerning a broad and interesting class of processes. Let us try to indicate that (4.1) does provide such a class of processes.

Cumulants measure at least two important aspects of random variables; the extent of non-normality of the random variables and the extent of joint dependence of the random variables. In the discussion that follows we will be concerned with the latter aspect. We note that (4.1) implies, among other things, that $c_{a_1,\ldots,a_k}(v_1,\ldots,v_{k-1},0)$ is tending to zero as the $|v_j| \to \infty$. In other words values of the process well separated in time are tending to become statistically independent. The "span of dependence" of the process is not great and such things as discrete cosinusoids are excluded. If we are dealing with a linear process

$$X(t) = \sum_u h(t-u)\epsilon(u) \tag{4.2}$$

with the $\epsilon(u)$ independent random variables having finite moments of all orders, then (4.1) will be satisfied if

$$\sum |u|^{\ell} |h(u)| < \infty \tag{4.3}$$

and we again note the low span of dependence.

B. Models. One may give a simple interpretation of the time reversibility of a process in terms of k-th order spectra. A process $X(t)$ is said to be time reversible if the probability structure of $X(-t)$ is the same as that of $X(t)$. If this is the case we see that

$$c_{a_1,\ldots,a_k}(-t_1,\ldots,-t_k) = c_{a_1,\ldots,a_k}(t_1,\ldots,t_k) \tag{4.4}$$

and therefore the imaginary part of $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ is identically zero. Conversely, we see that if the imaginary parts of all higher order spectra are identically zero, then (4.4) is satisfied for all a_j and k and therefore if the process is determined by its moments, then it is time reversible. In the case of a stationary Markov process it is easy to see what reversibility amounts to. Reversibility means that the probability structure looks the same going forwards in time as it does going backwards in time. This implies that the backwards or adjoint transition probability operator of the Markov process must be the same as the forwards transition probability operator, that is the transition probability operator is self-adjoint. Conversely self-adjointness of the transition probability operator implies reversibility of the Markov process.

One can also give examples of situations in which the real part of $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ is identically zero. Suppose one has a time reversible process $\{Y_{a_1}(t), X_{a_2}(t),\ldots,X_{a_k}(t)\}$; consequently the imaginary part of the k-th order spectrum is zero. Now let $X_{a_1}(t)$ be derived from $Y_{a_1}(t)$ by means of a linear time invariant operation

whose transfer function is purely imaginary. We see from (3.26) that the real part of $f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_k)$ is zero. In the case of a single time series we may make certain of the higher order spectra have real part zero. For example, let $Y(t)$ be a time reversible series and consider $X(t)$ which is derived from $Y(t)$ by a linear operation, again with purely imaginary transfer function. The real parts of all higher order spectra of odd orders of $X(t)$ will be identically zero. It is of course impossible, as is seen from the 2-nd order or power spectrum, for a single series to have the real parts of all spectra identically zero.

Let us now turn to the presentation of the k-th order spectra of a number of processes whose probability structure has been specified. We begin by noting an advantage of the definition of the k-th order spectrum as the Fourier transform of a cumulant. If the processes $X(t)$ and $Y(t)$ are statistically independent of one another, then the cumulants of $Z(t) = X(t) + Y(t)$ are the sum of the corresponding cumulants of $X(t)$ and $Y(t)$. It therefore follows that a k-th order spectrum of components of $Z(t)$ is simply the sum of the corresponding k-th order spectra of $X(t)$ and $Y(t)$.

Consider a stochastic process $X(t)$, $t = 0, \pm 1, \pm 2, \ldots$ of independent identically distributed random variables whose cumulants of all orders exist, the k-th being κ_k. The cumulant function of order k of this process is given by

$$c(t_1,\ldots,t_k) = \begin{cases} \kappa_k & \text{if} \quad t_1 = t_2 = \ldots = t_k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.5}$$

We see that the k-th order spectrum is given by

$$f(\lambda_1,\ldots,\lambda_k) = (2\pi)^{-k+1}\kappa_k \tag{4.6}$$

if $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$. In other words the real part is constant while the imaginary part is identically zero. If we now consider the linear process $Y(t) = \Sigma_u h(t-u)X(u)$ derived from the above $X(t)$ we see from (3.26) that its k-th order spectrum is given by

$$(2\pi)^{-k+1}\kappa_k \, H(\lambda_1)\ldots H(\lambda_k) \tag{4.7}$$

where $\Sigma_1^k \lambda_j \equiv 0 \pmod{2\pi}$ and $H(\lambda)$ denotes the Fourier transform of $h(u)$. As a byproduct of (4.7) we note that the coefficient

$$f(\lambda_1, \ldots, \lambda_k) / \{f(\lambda_1) \ldots f(\lambda_k)\}^{1/2} \tag{4.8}$$

will be constant, ($f(\lambda)$ denoting the 2-nd order spectrum).

We now turn to a simple model of a stationary process with a discrete spectrum. The model yields a vector-valued process with discrete spectrum; unfortunately it is not clear how this model might be meaningfully generalized to obtain a process with continuous spectrum. Let

$$X_s(t) = \sum_j Z^{(s)}(\lambda_j) \exp\{it\lambda_j\} = \sum_j R^{(s)}(\lambda_j) \exp\{it\lambda_j + i[\theta(\lambda_j) + \tau^{(s)}(\lambda_j)]\}, \tag{4.9}$$

$s = 0, 1, \ldots, m$ be a set of $m+1$ real-valued, stationary, random processes with jumps in the spectrum at λ_j. The real-valued character of the process implies that if λ_j is in the spectrum, so is $-\lambda_j$ with

$$Z^{(s)}(-\lambda_j) = \overline{Z^{(s)}(\lambda_j)} . \tag{4.10}$$

We shall explicitly assume that

$$R^{(s)}(\lambda_j) = |Z^{(s)}(\lambda_j)| > 0 . \tag{4.11}$$

We set

$$\theta(\lambda_j) + \tau^{(s)}(\lambda_j) = \arg Z^{(s)}(\lambda_j) \tag{4.12}$$

with $\theta(\lambda_j)$, $\lambda_j \geq 0$, independent random variables uniformly distributed on $[0, 2\pi]$. Further the set of $\theta(\lambda_j)$ are assumed to be independent of all the $R^{(s)}(\lambda_j)$, but the $R^{(s)}(\lambda_j)$ among themselves can have any dependent structure. The object is to introduce a model for which some meaning can be attributed to relative time lags for harmonics of different components of the vector-valued process. Notice that, for example,

$$E\{[Z^{(s_1)}(\lambda_{j_1})]^{\alpha_1}[Z^{(s_1')}(-\lambda_{j_1})]^{\beta_1} \ldots [Z^{(s_a)}(\lambda_{j_a})]^{\alpha_a}[Z^{(s_a')}(-\lambda_{j_a})]^{\beta_a}\} \tag{4.13}$$

$$= \delta_{\alpha_1-\beta_1} \ldots \delta_{\alpha_a-\beta_a} E\{[R^{(s_1)}(\lambda_{j_1})R^{(s_1')}(\lambda_{j_1})]^{\alpha_1} \ldots [R^{(s_a)}(\lambda_{j_a})R^{(s_a')}(\lambda_{j_a})]^{\alpha_a}\}$$

if $0 < \lambda_{j_1} < \ldots < \lambda_{j_a}$, and we see that we may derive expressions for the joint cumulants of the process. We see that the k-th order spectra are sums of Dirac delta functions.

As a final example suppose that we have a stationary Gaussian series $X(t)$, $t = 0, \pm 1, \ldots$ with $E X(t) = 0$, $E X(t) X(s) = r(t-s)$, and power spectral density $f(\lambda)$. Suppose

$$Y(t) = \sum_u A(u) X(t-u) + \sum_{u_1, u_2} B(u_1, u_2) X(t-u_1) X(t-u_2) \tag{4.14}$$

where the Fourier transforms of A and B are a and b respectively, then carrying out elementary computations we see

$$E Y(t) X(t-s) = \int \exp\{-is\lambda\} a(\lambda) f(\lambda) d\lambda \tag{4.15}$$

$$E Y(t) X(t-s_1) X(t-s_2) = r(s_1-s_2) \int b(\lambda, -\lambda) f(\lambda) d\lambda \tag{4.16}$$

$$+ 2 \int\int b(\lambda_1, \lambda_2) \exp\{-i(s_1\lambda_1 + s_2\lambda_2)\} f(\lambda_1) f(\lambda_2) d\lambda_1 d\lambda_2$$

$$E Y(t) = \int b(\lambda, -\lambda) f(\lambda) d\lambda \tag{4.17}$$

$$\operatorname{cov}\{Y(t), Y(t-s)\} = \int |a(\lambda)|^2 \exp\{-is\lambda\} f(\lambda) d\lambda \tag{4.18}$$

$$+ 2 \int\int |b(\lambda_1, \lambda_2)|^2 \exp\{-is(\lambda_1 + \lambda_2)\} f(\lambda_1) f(\lambda_2) d\lambda_1 d\lambda_2$$

and the 3-rd order spectrum is

$$2[a(\lambda_1)a(\lambda_2)b(-\lambda_1, -\lambda_2) f(\lambda_1)f(\lambda_2) + a(\lambda_2) a(\lambda_3) b(-\lambda_2, -\lambda_3) f(\lambda_2) f(\lambda_3) \tag{4.19}$$

$$+ a(\lambda_3) a(\lambda_1) b(-\lambda_3, -\lambda_1) f(\lambda_3) f(\lambda_1)]$$

$$+ 8 \int \overline{b(\lambda, \lambda_1-\lambda) b(\lambda_2+\lambda, -\lambda) b(\lambda-\lambda_1, -\lambda-\lambda_2) f(\lambda_1-\lambda) f(\lambda_2-\lambda) f(\lambda)} d\lambda$$

where the bar indicates that all permutations of $\lambda, \lambda_1, \lambda_2$ are averaged.

C. Effects of Second Order Spectral Mass. Given the time series $X(t) = (X_a(t), a = 1, \ldots, r)$ one feels intuitively that if for some series, say $X_{a_1}(t)$, and frequency λ_1, $f_{a_1 a_1}(\lambda_1, -\lambda_1)$ is negligible, then all higher order spectra involving the series $X_{a_1}(t)$ at the frequency λ_1 should also be negligible. One feels this since the second order spectrum is a measure of the presence of a harmonic component. We may make this statement precise as follows: given random variables $y_1, \ldots, y_k$, from Schwartz's Inequality we see that

$$(4.20) \qquad |E y_1 \cdots y_k| \leq \{E y_1^2 \, E| y_2 \cdots y_k|^2\}^{1/2}$$

provided the moments involved exist. If we set $y_j = Z_{a_j}(\lambda_j + \Delta\lambda_j) - Z_{a_j}(\lambda_j)$ where $Z_{a_j}(\lambda)$ is the spectral function of (2.11) and $\Sigma_1^k \lambda_j \equiv 0 \,(\mathrm{mod}\, 2\pi)$, but $(\lambda_1, \ldots, \lambda_k)$ does not lie in any proper submanifold, then (4.20) becomes

$$(4.21) \quad |f_{a_1, \ldots, a_k}(\lambda_1, \ldots, \lambda_k) \Delta\lambda_1 \ldots \Delta\lambda_{k-1}|$$

$$\leq \{f_{a_1, a_1}(\lambda_1, -\lambda_1)\Delta\lambda_1\}^{\frac{1}{2}} \{f_{a_2, a_2}(\lambda_2, -\lambda_2) \ldots f_{a_k, a_k}(\lambda_k, -\lambda_k) |\Delta\lambda_2 \ldots \Delta\lambda_{k-1}|$$

$$+ \text{ terms of lower order }\}^{\frac{1}{2}}$$

$$\leq \{\prod_1^k f_{a_j, a_j}(\lambda_j, -\lambda_j)\}^{\frac{1}{2}} \{|\Delta\lambda_1 \ldots \Delta\lambda_{k-1}|\}^{\frac{1}{2}} + \text{ terms of lower order }.$$

In other words, if we take account of the $\Delta\lambda$'s then $f_{a_1, \ldots, a_k}(\lambda_1, \ldots, \lambda_k)$ is bounded by a multiple of $f_{a_1, a_1}(\lambda_1, -\lambda_1)$, and we see that if $f_{a_1, a_1}(\lambda_1, -\lambda_1)$ is zero in a neighborhood of λ_1, then the $f_{a_1, \ldots, a_k}(\lambda_1, \ldots, \lambda_k)$ will also be zero in that neighborhood. However we may not let the $\Delta\lambda$'s $\to 0$ and obtain a useful point result unless $k = 2$. If the λ's are in some proper submanifold, because the spectra involved are continuous functions in the whole manifold, we may consider a sequence of λ's approaching the submanifold and we see that (4.21) holds in a submanifold as well.

V. A Worked Example.

A. A Description of Sunspot Numbers. In Section V we record the results of estimating the 2-nd, 3-rd and 4-th order spectra of the classical series of sunspot numbers. We have chosen this series for analysis as it has a certain historic interest for statisticians, see [1], [10] and [11], and long stretches are now available. In fact, the series is obtainable on a daily basis from 1818 to 1965 and on a monthly basis from 1749 to 1965, (see [9]). The calculations reported are restricted primarily to the series of monthly averages; however we will comment briefly on the effects of using monthly rather than daily data.

Daily relative sunspot numbers are based upon counts of spots and group entities of spots on the sun's surface at some time each day, see [4] or [9]. A basic difficulty involved in the collection of such data lies in the fact that the number of spots and groups seen depends on the particular observer and his telescope. Wolf devised a formula, yielding a relative number, with the intent of reducing the spot counts of different observers and telescopes to a common basis. He defined his relative sunspot number, r, for the day as

$$(5.1) \qquad r = k(f + 10g)$$

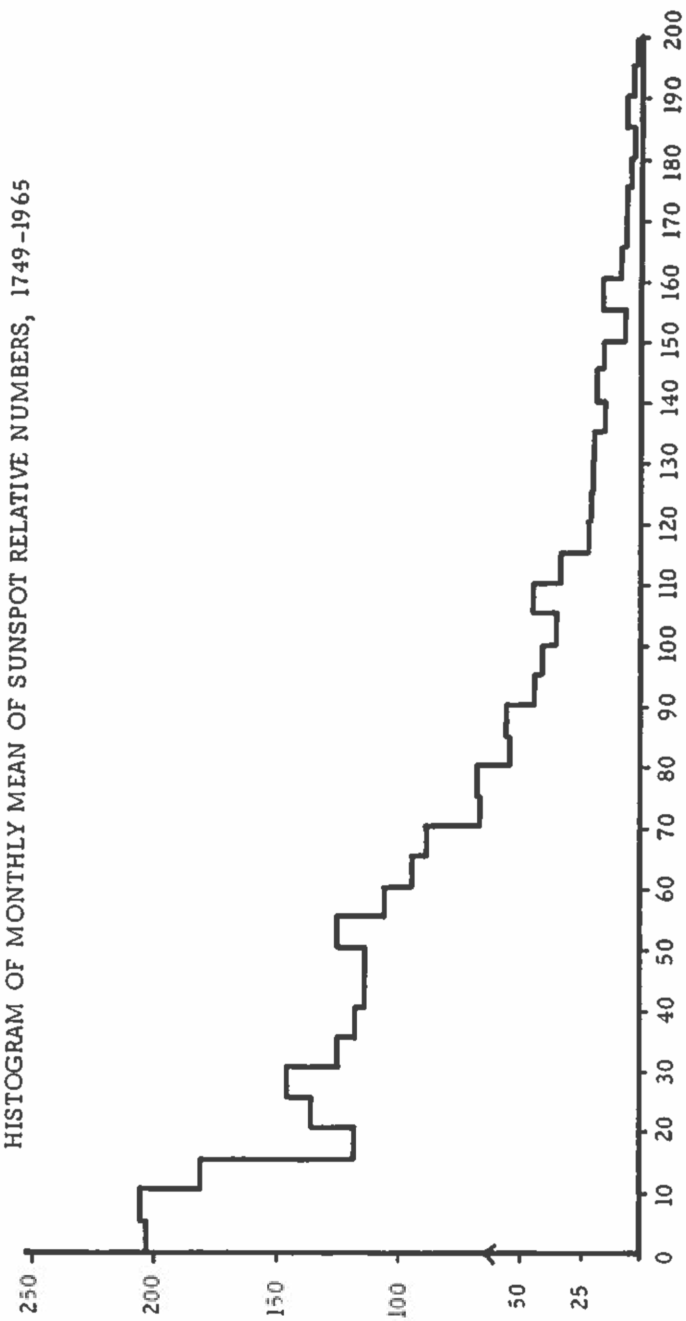
where, for a given day, g is the number of groups, irrespective of the number of component spots, f is the total number of component spots which can be counted in these groups (this may range from 1 to 50 or more in the case of complex groups) and k is a factor depending on the estimated efficiency of the observer and his telescope.

Turning to the series itself, it was suggested by Schwabe in 1843 that the series has a cycle of approximately 11 years. If one examines a graph of the series, see [5] or [9], the cyclic nature is readily apparent. However over the years the period has varied between 10 and 13 years. If the series is complex demodulated, at a period of 11 years, this apparently unsystematic variation in period becomes even more evident. (See [6] for a description of the technique of complex demodulation.) It has been suggested that two consecutive cycles should in fact be regarded as the two halves (one in reverse direction) of a cycle of approximately 22 years and that this latter period represents the true cycle of solar events. No overly compelling theories have been put forward to explain this 11 or 22 year cycle; however its effects are felt on the earth in a variety of ways.

Another aspect of the series that becomes apparent from an examination of a graph is the non-cosinusoidal nature of the repeating waveform. The wave tends to rise quickly and fall slowly and also to remain longer at low values than high. This last is apparent in Figure 8, an estimated histogram of the monthly means of relative sunspot numbers. The distribution has a point of condensation at zero, (that is a number of observations are exactly zero), a mode

FIGURE 8

HISTOGRAM OF MONTHLY MEAN OF SUNSPOT RELATIVE NUMBERS, 1749-1965



The arrow at 0 indicates a point of condensation.

near zero and falls off rapidly with increasing values thereafter. We find a mean of 48.85, standard deviation of 42.23, and standardized third to sixth order moments of 1.2, 4.4, 12.3 and 42.8 respectively. Within the Pearson system of curves this leads to a Type I curve in view of the rapidity of fall-off. Unpublished research of Quenouille's has indicated that the square roots of the numbers should be nearer to normality than the actual numbers or their logarithms. The histogram of the square roots is presented in Figure 9.

B. The Second Order Case. Yule, [11], estimated the periodogram of a graduated series of annual mean sunspot numbers. Bartlett, [1] carried out a smoothing of this periodogram. We have carried out a second order analysis of the monthly numbers employing the techniques described in these papers and also using Fourier transforms of the sample autocovariance function with convergence factors. The results of these distinct methods of estimation are in close agreement. Figures 10 and 11 record the results of periodogram smoothing with two bandwidths. On examination of Figure 11 we note the presence of a peak in the neighborhood of 0.015π, an accompanying harmonic at 0.030π and a gradual fall-off with increasing frequency. The frequency 0.015π corresponds to a period of approximately 11 years as was to be expected. In Figure 11, and especially Figure 10 we note the presence of a broad hump in the region of 0.16π. A period of 1 year corresponds to a frequency of 0.167π; however (see Section V. F) 0.16π is an alias of the synodic period of 27 to 28 days. An analysis of daily data (see Section F) indicates that this last is probably not an important source of power however.

We have also carried out a second order analysis of the square roots of the monthly numbers. The resulting spectral estimate is given by the lower curve of Figure 11. We note that it generally follows the spectrum of the untransformed series at a lower level and that we experience an improved signal to noise ratio.

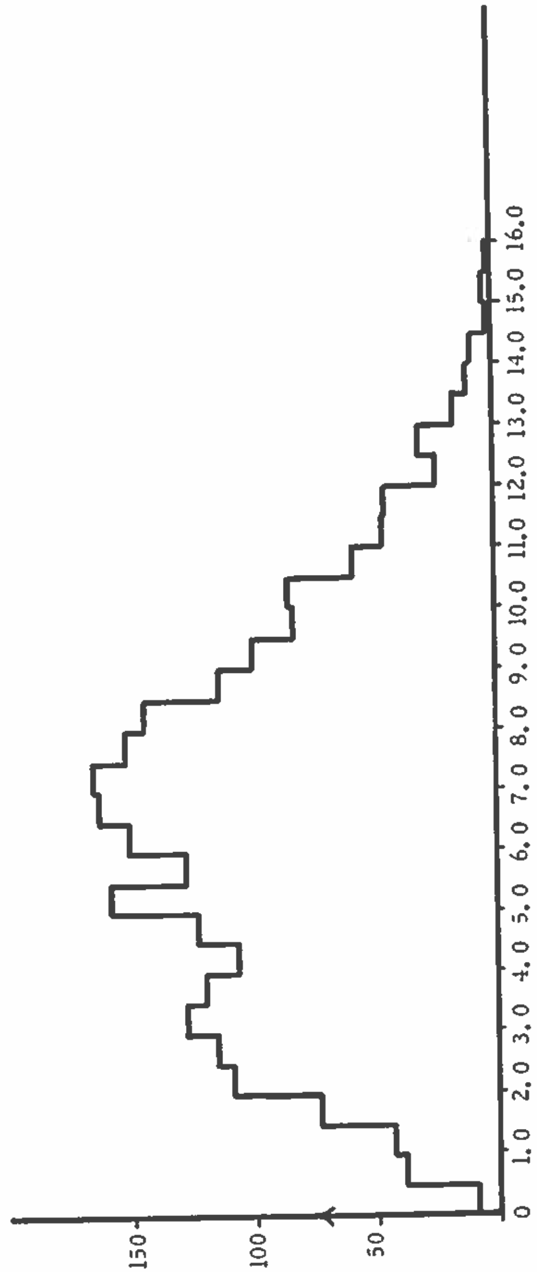
The weight function employed is given by (3. 48) and the asymptotic standard error calculated from (3. 49).

C. The Third Order Case. The results of a third order spectral analysis are given in Tables 1 - 4. The numbers presented are in fact estimates of the modulus and argument of the standardized coefficient, (3. 42), at varying frequencies. When a broad bandwidth is employed in the estimation, Tables 1 and 2, we note a vague constancy of the values, indicating the possibility of a linear process (see Section IV). There is some evidence of a ridge along $\lambda_2 = 0$; however this axis is one of the submanifolds of [3] and the ridge may simply be a manifestation of the increased variability. Turning to Table 2 specifically, if the process involved were time reversible, then the argument would be 0 or π for all frequencies. We do not appear

FIGURE 9

HISTOGRAM OF SQUARE ROOT MONTHLY MEAN OF SUNSPOT RELATIVE NUMBERS

1749 - 1965



The arrow at 0 indicates a point of condensation.

FIGURE 10. LOGARITHM OF ESTIMATED POWER SPECTRUM (MONTHLY NUMBERS)

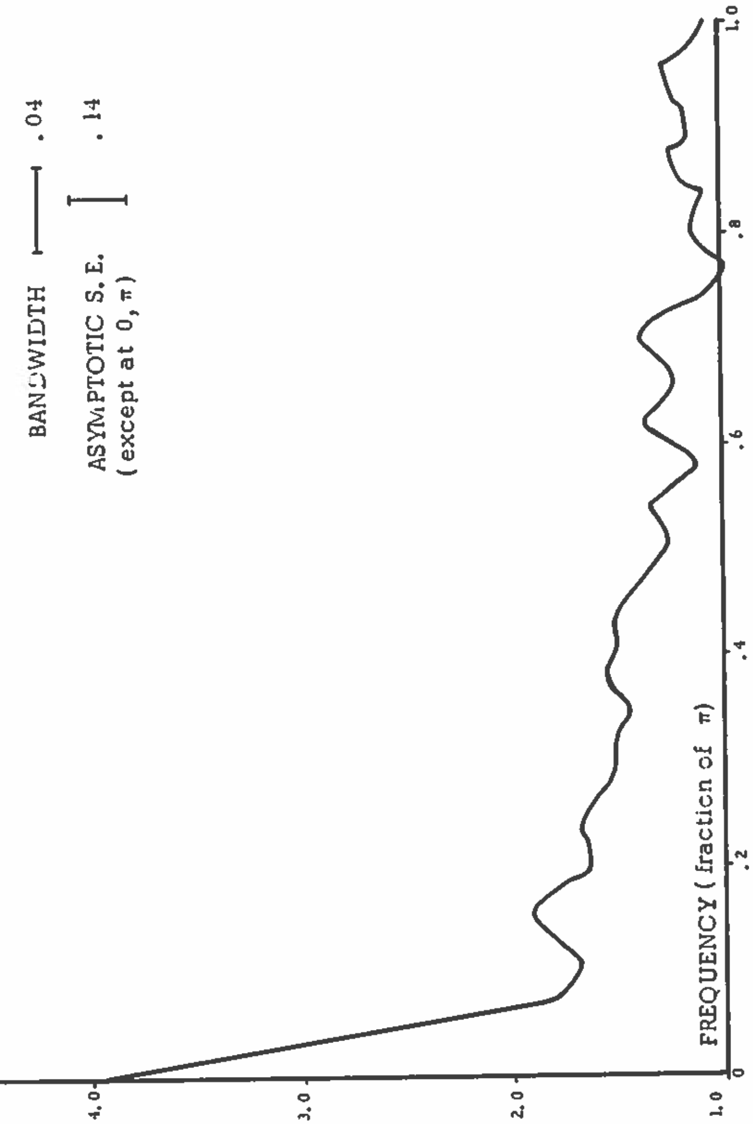
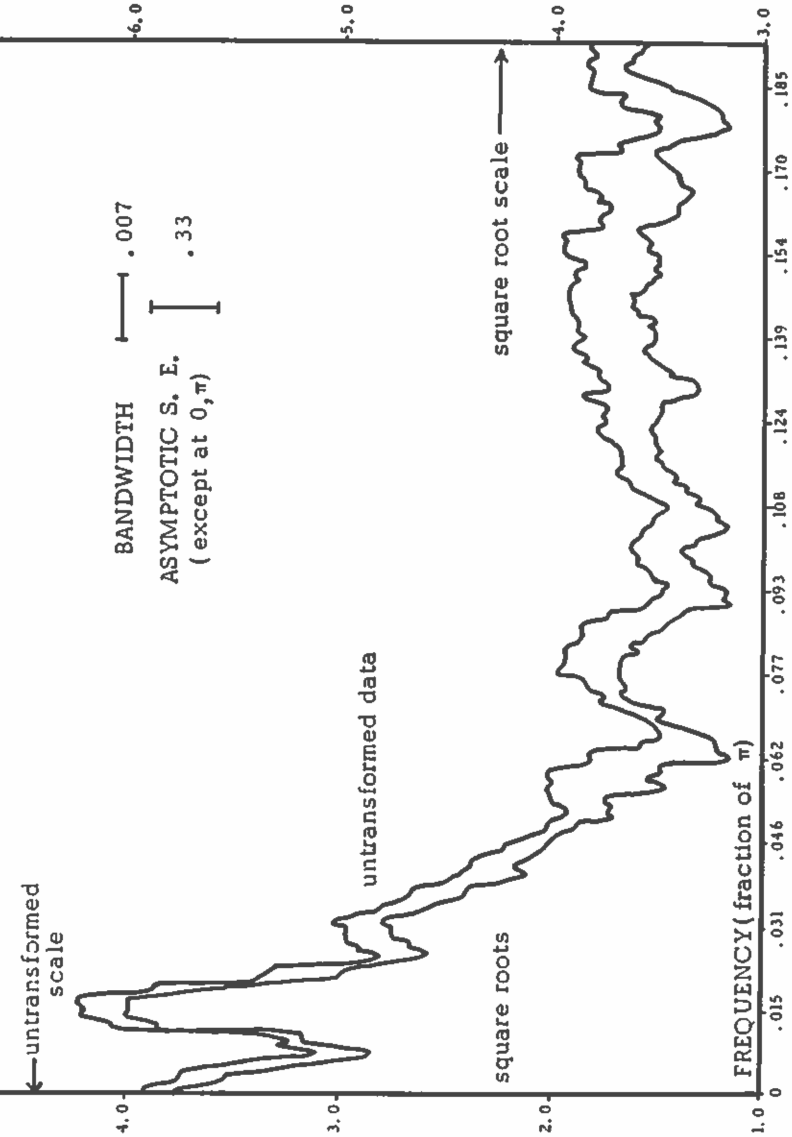


FIGURE 11. LOGARITHM OF ESTIMATED POWER SPECTRUM
(MONTHLY NUMBERS)
(SQUARE ROOT MONHTLY NUMBERS)



to have any evidence that this is the case. A narrower bandwidth is employed in the estimates of Tables 3 and 4. We now note the possibility of a peak in the neighborhood of $(\lambda_1, \lambda_2, \lambda_3) = (.015\pi, .015\pi, -.030\pi)$, that is the fundamental 11 year period and its first harmonic. This last is perhaps an indication of the non-cosinusoidal nature of the basic waveform.

Several remarks are in order concerning the calculations. As a result of symmetries, the imaginary part of the population third order spectrum is zero along $\lambda_2 = 0$. The estimated third order spectrum does not have this property because the weight function (3.48) does not possess the required symmetries. This will also contribute to the ridge at $\lambda_2 = 0$. The asymptotic standard errors provided in the Tables were calculated from formula (3.49). Elementary calculations indicate that these numbers in fact provide upper bounds for the required values since for f complex, var $|f| \leq$ cov $\{f, f\}$.

D. <u>The Fourth Order Case.</u> Tables 5 and 6 record a number of estimates of the standardized fourth order spectrum in a neighborhood of zero. The frequencies selected are those, at a constant spacing, nearest to zero in Figure 7. The submanifolds in these tables are the line $\lambda_2 = 0$, $\lambda_3 = 0$ and the plane $\lambda_2 + \lambda_3 = 0$.

E. <u>Some Details of the Computations.</u> The data analysed were, in fact, the monthly sunspot numbers from the year 1750 to the year 1965. We dropped the first year in order to have $2592 = 2^5 3^4$ months of data and therefore be able to usefully employ the Fast Fourier Transform Algorithm, [7]. The data had a linear trend removed and then the first and last 10% were tapered prior to the Fourier analysis. As an aid to casual perusal and in order to simplify the complex arithmetic of later computer programs, the complex Fourier coefficients were transformed to polar coordinates. In order to deal with the problem of smoothings involving points outside the fundamental regions, the Fourier transforms were numerically extended outside the fundamental range in a manner consistent with the symmetries involved. This augmented series was treated as the basic statistic under analysis.

The calculations were carried out on the University of London Atlas which has an addition time of 1.6μ sec. and a multiplication time of 6μ sec. A selection of computing times experienced include; Tables 1 and 2, 36 seconds from the Fourier transform and Tables 5 and 6, 11 minutes from the Fourier transform.

F. <u>Daily Sunspot Numbers.</u> An analysis of the daily values from 1957 to 1965 has indicated that the histogram has the same basic shape as that of Figure 8. The autocovariance function has humps at 27 1/4 days and 54 1/2 days; however the power spectrum has only

TABLE 1

ESTIMATED STANDARDIZED BISPECTRAL MODULUS

BANDWIDTH .04

ASYMPTOTIC S. E. .4
(except in submanifolds)

λ_2

																5.25									
															6.44	1.80	3.06								
														4.14	3.59	2.43	.20								
													.60	1.67	1.77	1.38	2.59	1.87							
												6.1	4.20	2.68	1.76	.50	4.52	2.01							
											6.24	3.2	6.60	2.49	1.96	1.43	3.78	2.63	1.24						
										.80	.30	2.0	.65	2.01	1.24	.88	.73	.49	3.48						
									2.30	6.41	6.59	2.8	1.69	1.89	3.41	2.69	4.78	.96	4.49	3.02					
								5.15	3.62	4.95	3.81	1.3	1.42	3.36	1.21	4.71	7.29	3.37	2.93	2.13	1.38				
							1.62	2.88	3.90	6.29	.91	2.2	2.10	1.28	1.41	1.89	1.35	2.36	2.63	3.34	3.91				
						3.37	1.39	5.24	2.94	2.72	.67	3.9	2.93	2.35	.12	4.93	1.05	1.37	1.43	1.75	3.12	1.66			
					7.83	3.05	3.07	4.20	6.19	3.12	5.71	2.1	1.34	1.67	3.81	1.23	3.37	2.14	2.68	2.37	1.82	.76			
				5.18	1.09	3.74	2.81	1.68	2.99	3.50	.62	2.4	.99	6.03	1.52	2.31	.21	4.44	3.59	1.72	3.07	1.35	1.89		
			2.87	2.79	4.47	3.67	4.28	4.04	4.33	5.30	2.97	2.6	2.10	3.27	.75	.54	2.70	2.06	3.70	1.22	3.18	3.14	1.94		
		7.75	2.36	2.92	5.07	2.03	4.34	1.35	2.30	2.51	3.51	5.5	2.84	2.67	2.37	3.70	1.88	1.61	3.06	3.89	2.22	3.13	.83	.76	
	11.67	3.98	5.30	5.10	1.72	5.17	5.39	3.72	5.49	3.27	6.45	6.7	3.07	2.69	1.73	1.87	4.62	5.32	4.34	4.18	2.07	2.37	1.26	7.22	
5.46	15.82	11.89	8.09	10.57	10.25	4.83	8.18	7.13	6.28	10.58	8.53	8.8	5.77	7.04	8.30	7.53	9.97	7.89	6.68	7.44	5.97	7.05	9.56	8.43	8.51
.0					.2					.4					.6					.8					1.0

λ_1 (AS A FRACTION OF π)

TABLE 2

ESTIMATED BISPECTRAL ARGUMENT

																	5.52									
																2.80	4.59	2.69								
															.85	1.13	.89	1.43								
														3.07	4.21	1.37	5.36	4.60	3.46							
													.19	6.24	4.38	2.03	1.47	5.13	3.87							
												6.01	5.57	6.08	5.19	.82	.71	5.97	1.27	5.14						
											6.02	1.98	1.90	4.44	.50	.58	.92	4.77	2.65	1.39						
										.08	.16	4.11	2.65	3.43	1.80	1.67	.22	4.54	6.20	1.17	1.65					
									6.04	5.63	6.03	4.39	3.29	5.82	5.24	1.40	.56	1.53	1.94	.79	.57	3.98				
								4.87	4.48	5.37	.86	.14	.01	5.89	1.44	.39	3.84	5.69	3.23	2.31	4.53	.39				
λ_2							.52	6.24	6.11	6.24	.12	3.49	.60	1.47	4.38	1.45	3.82	5.70	2.58	1.01	.73	5.47	.37			
						1.54	1.80	1.77	.57	.61	5.33	5.47	5.90	3.16	.85	1.66	2.13	1.28	.92	2.28	1.71	5.29	5.58			
					6.26	1.64	1.27	3.99	1.45	.87	1.38	4.70	.50	1.91	4.22	4.38	2.67	.96	.50	.49	.43	1.11	3.72	4.81		
				1.48	.52	1.28	.37	6.17	.89	5.70	6.21	.20	4.67	.87	5.70	2.18	4.06	4.48	1.87	1.04	6.04	2.15	3.06	4.47		
			1.03	6.23	4.29	2.99	1.33	.03	.83	4.26	.26	6.05	5.34	3.66	1.59	1.93	3.37	3.84	.05	2.24	.70	2.22	6.01	.31	.96	
		.99	.68	.00	6.18	.36	5.68	.62	6.21	.06	.46	6.15	.88	.49	.34	5.92	6.24	.00	5.93	6.14	5.70	1.21	5.56	5.65	6.11	
	.00	5.63	6.14	.07	6.20	.00	6.22	.03	6.28	6.28	6.26	.11	6.28	6.22	6.23	.01	.03	.08	.00	6.02	6.21	.17	.02	6.15	.16	6.26

λ_1 (AS A FRACTION OF π)

	.0					.2					.4					.6					.8					1.0

TABLE 3

ESTIMATED STANDARDIZED BISPECTRAL MODULUS

BANDWIDTH .007

ASYMPTOTIC S. E. 2.2
(except in submanifolds)

											17.07
										13.32	14.23
									18.60	4.62	4.49
								16.18	13.88	2.95	3.56
							23.42	9.68	9.02	23.17	18.57
λ_2						34.09	24.95	4.02	12.83	27.61	24.22
					24.13	22.19	20.74	9.73	14.58	15.75	15.35
				7.72	18.14	12.40	12.56	6.95	8.07	4.19	7.96
			15.53	4.10	8.09	6.72	12.69	15.63	12.02	3.25	8.85
		13.31	12.57	5.47	13.78	21.48	12.54	9.19	14.85	22.52	13.02
	7.94	5.82	9.57	6.37	9.09	22.02	19.56	4.67	12.72	22.73	23.74

λ_1 (AS A FRACTION OF π)

.0 .02 .04

TABLE 4

ESTIMATED BISPECTRAL ARGUMENT

											6.02
										.63	5.95
									6.03	.02	5.89
								.31	.07	5.83	4.26
							1.62	.35	.95	1.25	1.13
λ_2						1.33	1.58	.39	.87	1.45	1.05
					1.25	1.22	1.29	1.41	1.11	1.27	1.05
				1.05	1.20	1.26	.93	.90	.35	6.26	.60
			5.22	.38	.74	5.66	5.33	.48	.91	2.60	3.82
		4.39	4.71	6.22	.52	.15	5.94	.27	.17	.32	6.12
	3.14	4.39	5.37	.99	.21	6.07	6.13	6.20	6.24	6.27	.09

λ_1 (AS A FRACTION OF π)

.0 .02 .04

ESTIMATED STANDARDIZED TRISPECTRAL MODULUS

TABLE 5-a $\lambda_1 = .00$; $\lambda_2 = .00$; $\lambda_3 = .00$

3.39

TABLE 5-b $\lambda_1 = .04$

	$\lambda_3 = -.04$	.00	.04
$\lambda_2 = .04$	19.64	15.39	5.60
.00		11.51	

TABLE 5-c $\lambda_1 = .08$

	$\lambda_3 = -.08$	-.04	.00	.04	.08
$\lambda_2 = .08$	23.62	5.06	5.19	6.63	.95
.04		7.58	4.37	4.12	
.00			7.62		

BANDWIDTH .04

ASYMPTOTIC S. E. 1.2
(except in submanifolds)

ESTIMATED TRISPECTRAL ARGUMENT

TABLE 6-a $\lambda_1 = .00$; $\lambda_2 = .00$; $\lambda_3 = .00$

3.12

TABLE 6-b $\lambda_1 = .04$

	$\lambda_3 = -.04$	.00	.04
$\lambda_2 = .04$	.21	.46	1.84
.00		5.11	

TABLE 6-c $\lambda_1 = .08$

	$\lambda_3 = -.08$	-.04	.00	.04	.08
$\lambda_2 = .08$	6.27	.47	.99	1.76	1.78
.04		.30	.47	1.19	
.00			5.72		

a very slight peak at the corresponding period. A possible explanation of this last is the following; the synodic rotation period or apparent period of rotation of the sun as seen from the earth is generally given as 27 1/4 days. Now the actual time of appearance of a sunspot is to a degree unrelated to the time of appearance of other spots, yet once a spot has appeared it will remain two rotations or so. Therefore the autocovariance function will tend to have peaks but not the power spectrum. (The cepstrum is a different matter.) If one is in fact dealing with monthly averages, the fundamental alias of a period of 27 1/4 days is approximately 0.16π; however the series of monthly averages may be thought of as a smoothed and decimated version of the daily series and therefore a period of 27 1/4 days should have little power.

VI. Proofs.

A. Proof of Lemma 1. We note that

$$\hat{X}_a(t) = T^{-1}\sum_{s=0}^{T-1} d_a\left(\frac{2\pi s}{T}\right)\exp\{i2\pi st/T\}, \tag{6.1}$$

therefore

$$T^{-1}\sum_{t=0}^{T-1}\hat{X}_{a_1}(t+v_1)\dots\hat{X}_{a_k}(t+v_k) \tag{6.2}$$

$$= T^{-k-1}\sum_{t=0}^{T-1}\sum_{s_1=0}^{T-1}\dots\sum_{s_k=0}^{T-1}\exp\{i\frac{2\pi}{T}(s_1v_1+\dots+s_kv_k)\}$$

$$\cdot\exp\{i\frac{2\pi t}{T}(s_1+\dots+s_k)\}\, d_{a_1}\left(\frac{2\pi s_1}{T}\right)\dots d_{a_k}\left(\frac{2\pi s_k}{T}\right)$$

giving (3.8). We note that

$$\eta_s = T^{-1}\sum_{t=0}^{T-1}\exp\{i\frac{2\pi ts}{T}\} \tag{6.3}$$

therefore (3.11) follows from (3.10) and the fact that

$$\Phi\left(\frac{2\pi s_1}{T},\dots,\frac{2\pi s_k}{T}\right) = \sum_{\nu}(-1)^{p-1}(p-1)!\,\eta_{\tilde{s}_{\nu_1}}\dots\eta_{\tilde{s}_{\nu_p}} \tag{6.4}$$

where the summation extends over all partitions $(\nu_1,\dots,\nu_p)$ of

$(1,\dots,k)$ and $\tilde{s}_\nu = s_{i_1}+\dots+s_{i_j}$ if $\nu = (i_1,\dots,i_j)$. The Fourier relation inverse to (3.11) is

$$J^{(T)}_{a_1,\dots,a_k}\left(\frac{2\pi s_1}{T},\dots,\frac{2\pi s_k}{T}\right)\Phi\left(\frac{2\pi s_1}{T},\dots,\frac{2\pi s_k}{T}\right) \tag{6.5}$$

$$= \sum_{v_1=0}^{T-1}\dots\sum_{v_k=0}^{T-1}\hat{c}_{a_1,\dots,a_k}(v_1,\dots,v_k)\exp\{-i\frac{2\pi}{T}(s_1v_1+\dots+s_kv_k)\}$$

from which (3.12) follows since

$$\hat{c}_{a_1,\dots,a_k}(t+v_1,\dots,t+v_k) = \hat{c}_{a_1,\dots,a_k}(v_1,\dots,v_k) \tag{6.6}$$

for $t = 0, \pm 1, \pm 2, \dots$.

VII. Acknowledgement.

The calculations reported in this paper were carried out on the London School of Economics' IBM 1440 and the University of London Atlas. We would like to thank C. E. Rogers of the London School of Economics for writing all the computer programs required and seeing the calculations involved through to completion.

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Computation and Interpretation of k-Th Order Spectra

I. Introduction and Summary of Results.

In this paper we turn to various aspects of the computation and interpretation of k-th order spectra. The effects of symmetries and the directly related problem of aliasing are discussed. In addition, we discuss the computational and algebraic properties of the estimate of a k-th order spectrum proposed in [3]. Brief discussions of models, pre-filtering, and band-pass filtering are included. We return to the classical series of sun-spot numbers [9] and compute estimates of the second, third and fourth order spectra.

II. Aspects of Symmetries.

A. Notation and Definitions. We will, to a certain extent, be commenting on results obtained in the accompanying paper on the asymptotic behavior of k-th order spectral estimates [3]. For this reason it will be convenient to adopt the notation employed there.

The process $X(t) = (X_a(t); a = 1, \dots, r)$ is assumed to be a stationary r-vector valued process with real-valued components. All moments of the process are assumed to exist and equivalently all cumulant functions are assumed to exist. The moment functions

$$(2.1) \quad m_{a_1,\dots,a_k}(t_1,\dots,t_k) = E X_{a_1}(t_1) \dots X_{a_k}(t_k)$$
$$= m_{a_1,\dots,a_k}(t+t_1,\dots,t+t_k)$$

and cumulant functions

$$(2.2) \quad c_{a_1,\dots,a_k}(t_1,\dots,t_k) = c(X_{a_1}(t_1),\dots,X_{a_k}(t_k))$$
$$= c_{a_1,\dots,a_k}(t+t_1,\dots,t+t_k)$$